Illiquidity of the annuity: a potential solution to the annuity puzzle

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Abstract

Annuity puzzle refers to the inconsistency between theoretical results and empirical data on annuity demand. In this paper, we construct a monetary general equilibrium dynastic model with money and annuities. There are two dimensions of an asset: return and liquidity. The bequest motive is an important factor that lowers liquidity of annuity. When the liquidity of annuities is too low, it would generate a theoretical result in which the annuity accounts for almost zero percent in the retirement wealth. A higher inflation rate reduces the value of money, and a stronger bequest motive reduces the liquidity of the annuity. Consequently, a higher bequest motive and a lower inflation rate reduce the demand for the annuity, which generates the theoretical result being consistent with empirical data.

JEL classification: E21; E41; E52

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1 Introduction

The problem of saving behavior is always presented in the theater of macroeconomists' screenplays. Saving is a means to shift income by diversified assets in order to smooth consumption intertemporally or, more importantly, arrange for the time with no labor income. Substantial studies have focused on how agents facing retirement allocate their wealth, and the most common background setting to deal with this issue is a life-cycle model with uncertainty of lifetime.[1] The annuity, a financial asset which ties down with the specific individual and provides returns every period until the holder dies, plays a critical role in retirement wealth.

The annuity makes its debut in the economic theoretical model by Yaari (1965), which points out that a risk-averse agent, facing longevity risk only and having no bequest motive with separable utility function setting, would choose to dispose all wealth in the annuity since it pays higher returns than the bond if the holder is alive. While the life-cycle model suggests that putting retirement wealth into the life annuity is optimal, the reality displays by no means in correspondence with the theoretical result. Modigliani (1986) is the first to sketch the inconsistency between theoretical results and empirical data in annuities, currently known as the annuity puzzle. Modigliani (1986) mentions: "[I]t is a well-known fact that annuity contracts, other than in the form of group insurance through pension systems, are extremely rare. Why this should be so is a subject of considerable current interest. It is still ill-understood." To understand the discrepancy between theoretical results and empirical data, imagine that there is a consol and an annuity in the market. Assume that prices of a consol and an annuity are identical. A bond guarantees periodic returns forever while the income stream of an annuity ends at the time when the annuitant dies. Under the no-arbitrage condition, the payment rate of an annuity should be higher than the interest rate

[1] The life-cycle model is able to characterize the hump-shape stylized lifetime income pattern and impose stochastic factors in any period.

of a consol.[2] That is, the annuity dominates the consol in the rate of return and should be attractive to retirees. Most theoretical results so far are consistent with the intuition mentioned above; however, the empirical data shows the opposite outcome: lack of voluntary annuitization. Echoing Modigliani (1986), Johnson et al. (2004) tabulates from Health and Retirement Study and shows that private annuities only account for one percent of total wealth.

The main contribution of this paper is to present a general equilibrium monetary model and generate results that are more consistent with facts. We develop the model based on Lagos and Wright (2005), in which the annuity is an alternative asset to money. The general equilibrium feature of the model allows us to determine endogenously the prices, the rate of return and liquidity of money and the annuity. We apply the idea of dynastic utility from Barro (1974) to capture the concept of altruism toward descendants. The dynastic model setting emphasizes that agents being generation after generation rather than a representative agent living forever. In the case that agents care about their children, they have the bequest motive.

The main features of the model are as follows. Each period is divided into three subperiods and resembles an agent's lifetime: working stage, young stage, and retirement stage. Unlike partial equilibrium models, we allow agents to work to accumulate wealth before retirement. This paper characterizes the behavior of insurance companies, hence endogenizes the price and the payout rate of the annuity. With the general equilibrium framework, we are able to clarify the factors that affect the benefits of money and annuities, and therefore characterize the demand for both assets. We show that the current model is capable of generating less or no demand for the annuity. The existence of survival shocks creates liquidity differentials between money and annuities: annuities are not available to finance consumption if the agent dies. The demand for the annuity would vanish if the payout rate could not compensate for illiquidity of the annuity. We present

[2] If an annuity offers \$5,000 per period as the holder is alive on a \$100,000 premium, the annuity payment rate is 5%.

conditions in each equilibrium and the analysis on the effect of the inflation rate and the degree of altruism.

In this model, we inject nominal asset, money, so that we can discuss the effect of the inflation rate since it influences the value of money and the annuity differently. Because forward-looking agents in the dynastic model take their children's utility into consideration, the higher inflation rate lowers the future value of money hence less consumption of their child through money. In the situation that agents partially annuitize their wealth, the higher inflation rate raises the annuity demand since money is less valuable. We also prove that when agents have the higher bequest motive in which they partially annuitize their wealth, it amplifies illiquidity of the annuity and agents annuitize less. In the dynastic model, as money always has the ability to finance children's consumption while the annuity loses the ability when the agent dies early, the stronger bequest motive causes relatively greater loss on the inability to finance children's consumption of the annuity because agents care more about their children. Agents prefer money to the annuity under the higher bequest motive, hence less demand for the annuity.

1.1 Literature Review

Since the discrepancy between theoretical results, in which people should annuitize a large fraction of their wealth, and empirical data, in which people barely annuitize their wealth, is obvious, macroeconomists attempt to make the theoretical result agree with the empirical data. Davidoff et al. (2005) analyze the optimal retirement portfolio problem by means of the dual method of utility maximization: expenditure minimization. By relaxing the assumption of separable utility function, full annuitization is still optimal with no bequest motive under a complete market. Even if agents have the bequest motive and face the medical expense risk in an incomplete market, the optimality still displays partial annuitization more than two-third of retirement wealth. Davidoff et al. (2005) also mention that the uninsured medical expense causes lower survival rate, hence lower demand for the annuity.

Economists also resort to psychological and behavioral explanations other than rational models. Brown (2007) states that private information of health conditions leading to higher annuity prices, pre-existing annuitization, risk sharing among family members, and bequest motives would reduce the demand for the annuity. Some possible psychological or behavioral factors such as thought of the annuity contract as a complex financial contract and the illusion of losing control of a long-term contract may influence the demand for the annuity. Besides, Brown et al. (2008) suggest that annuities are more attractive when people view them as consumption payments rather than investment earnings. Benartzi et al. (2011) mention that people might not save enough to buy annuities at retirement and many households live by Social Security rather than by annuity incomes. On the other hand, Poterba et al. (2011) argue that medical expense shocks and family structure shocks make the annuity less appealing because the liquidation value of annuities is low. Instead of annuities, housing equity serves as a precautionary asset. However, some economists still strive for the theoretically consistent result. Davidoff (2009) shows that long-term care insurance and the annuity are complementary since long-term care insurance could help extend one's life and make the annuity more attractive, but house equity crowds out long-term care insurance and the annuity because house equity accounts for a large proportion of wealth and retirees liquidate their house equity when they need long-term care. Yogo (2016) shows that stocks are positively correlated to health while health expenditure and share in housing are negatively correlated to health. Even considering an incomplete annuity market, bequest motives, background risk and default risk in a life-cycle model, annuitizing large fraction of wealth remains optimal in Peijnenburg et al. (2016), which leads to the title: "the annuity puzzle remains a puzzle."

Literatures so far could give explainations for "less" annuity demand, but not for "almost zero" annuity demand, except Lockwood (2012), which resorts to the bequest motive as the key factor for extremely low annuity demand. He considers a partial equilibrium life-cycle model to depict that sufficiently large loads on the annuity price would eliminate the annuity demand. In his model, the loads, wealth at age 65, and pre-existing annuity incomes are exogenously given. However, in reality, people could withdraw pension in a lump sum at age 65. Moreover, he considers only the decision to allocate wealth in consumption and bequest after retirement, but he ignores agents' forward-looking consideration before retirement to accumulate more wealth. Different from Lockwood (2012), who generates low annuity demand in a model where the annuity price and the fraction of pre-existing annuitization wealth are exogenously given, we provide a general equilibrium monetary model in which forward-looking agents endogenously choose their wealth accumulation and asset allocation, and all the prices and liquidity are endogenously determined. We are able to discuss how the inflation rate and the bequest motive affect the payout rate and the demand for the annuity, which are absent in Lockwood (2012).

The paper proceeds as follows. Section 2 specifies the model. Section 3 derives the equilibrium conditions from sellers' and buyers' decision problems. Section 4 discusses all equilibria and comparative static analysis. Section 5 provides some numerical results. Section 6 discusses the different effects of the inflation rate between this paper and a life-cycle partial equilibrium model. Section 7 concludes. All proofs and details are contained in the Appendix.

2 The Basic Model

The basic model follows Lagos and Wright (2005), Telyukova and Wright (2008), and Guerrieri and Lorenzoni (2009). Time is discrete and continues forever. Each period is divided into three subperiods, in which agents trade consumption goods in the frictionless centralized market.[3] All goods in each subperiod are nonstorable and perfectly divisible. There is a $[0,2]$ continuum of agents at the beginning of each period: the seller and the buyer account for a unit measure respectively. Buyers only consume and sellers only produce in the first subperiod and the second subperiod. All agents can consume and produce in the third subperiod.

2.1 Buyer

A buyer may be born in the second or third subperiod. They face an idiosyncratic survival shock, $\rho \in (0,1)$, in the beginning of the second subperiod. With probability ρ, a buyer survives in the second subperiod, and then dies at the end of the second subperiod and his offspring will be a new born in the third subperiod. With probability $1-\rho$, a buyer passes away at the beginning of the second subperiod. His child will catch up and be idle in the second subperiod, and start to work, consume and adjust portfolio in the third subperiod.[4] Let $\beta_i \in (0,1)$ be the discount factor between subperiod i and $i-1$ within a buyer's lifetime, where $i = 1,2$. The degree of altruism of a buyer is $\beta_a \in [0,1)$, which discounts the utility of children into that of the buyer. Combining conditions above, we have $\beta_1\beta_2\beta_a < 1$.

Buyers in subperiod i have utility $u_i(q_i)$ from q_i consumption, where $i = 1,2$. In the third subperiod, buyers produce, consume to get utility $U(X) - h$ from X goods and h labor hour, and adjust their asset holding of money and annuities. This is the standard quasi-linear utility function setting following Lagos and Wright (2005). Assume $U(0) = 0$, $U'(0) = \infty$, $U'(X) > 0$, $U''(X) < 0$, $u_i(0) = 0$, $u_i'(0) = \infty$, $u_i'(q_i) > 0$, $u_i''(q_i) < 0$, where $i = 1,2$. Suppose utility of a buyer's child

[3]Different from Lagos and Wright (2005), Telyukova and Wright (2008), and Guerrieri and Lorenzoni (2009), there is no decentralized market in this model.

[4]The idle child could be thought of as the babyhood, in which we assume that the child does not have the ability to involve in the market actions.

is V_c, the dynastic utility function of a buyer is

$$U(X) - h + \beta_1 u_1(q_1) + \beta_1 \beta_2 u_2(q_2) + \beta_1 \beta_2 \beta_a V_c.$$

Figure 1 shows the timeline of a buyer.

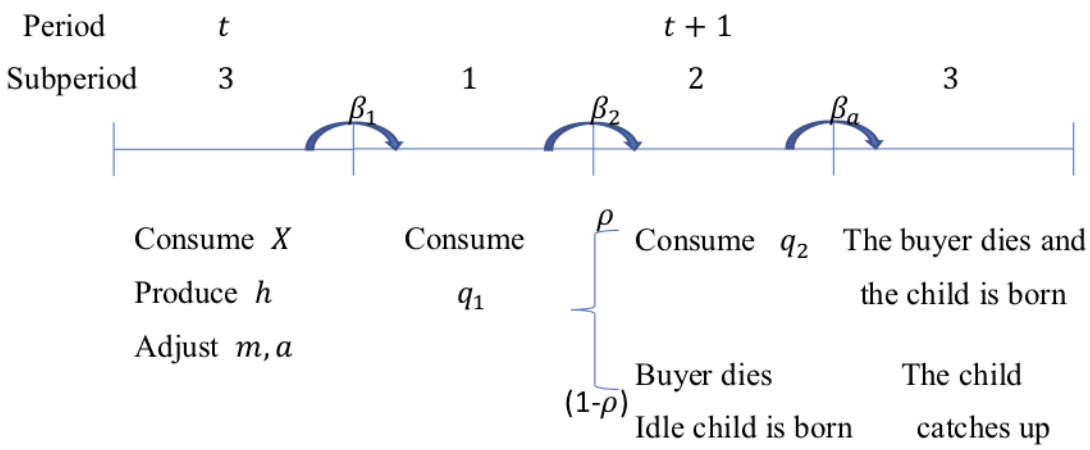

Figure 1: Timeline of Buyers

2.2 Seller

A seller lives for three subperiods certainly. They are born in the first subperiod and die at the end of the third subperiod. They produce and do not consume in the first subperiod and the second subperiod, but like buyers, a seller can produce and consume in the third subperiod. Let $\beta_i^s \in (0,1)$ be the discount factor between subperiods i and the last subperiod within a seller's lifetime, where $i = 2,3$. Let the degree of altruism of a seller be $\beta_a^s \in (0,1)$. Combining conditions above, we have $\beta_2^s \beta_3^s \beta_a^s < 1$.

Sellers incur disutility $c_i(q_i^s)$ from producing q_i^s units of consumption goods, where $i = 1,2$. Assume $c_i(0) = 0$, $c_i'(q_i^s) > 0$, $c_i''(q_i^s) \geq 0$. In the third subperiod, a seller gets utility $U(X^s) - h^s$ from X^s goods and h^s labor hour. Suppose utility of a seller's child is V_c^s, the dynastic utility function of a seller is

$$-c_1(q_1^s) - \beta_2^s c_2(q_2^s) + \beta_2^s \beta_3^s [U(X^s) - h^s] + \beta_2^s \beta_3^s \beta_a^s V_c^s.$$

Figure 2 shows the timeline of a seller.

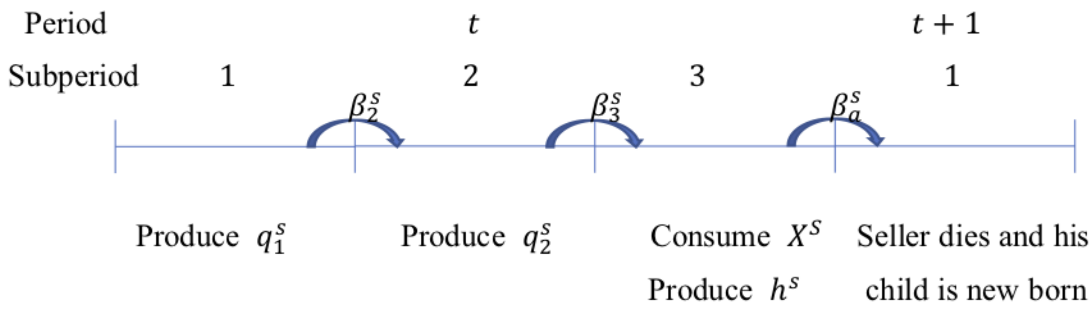

Figure 2: Timeline of sellers

We assume that all buyers and sellers do not own record-keeping and enforcement technology, so credit plays no role in the economy even though there are frictionless centralized markets; *i.e.*, every transaction must be *quid pro quo*. To accomplish transactions in the first two subperiods, buyers must carry assets in exchange for consumption goods. Without loss of generality, we assume $\beta_1 = \beta_2 = \beta_2^s = \beta_3^s = \beta_a^s = \beta$ in the following discussion.

The government is the sole issuer of money. money has no intrinsic value, yet it plays an essential role as a means of payment in the economy, more pecisely, in the first and second subperiod, where credit is not feasible. We assume money is perfectly divisible and storable. There is no storage cost or transaction cost when storing or using money. money stock evolution is determined exogenously at a gross rate γ by the government. That is, $M_{t+1} = \gamma M_t$. In the third subperiod, money is injected to buyers by lump-sum transfer if $\gamma > 1$, or withdrawn from buyers through taxation by the government if $\gamma < 1$. The aggregate transfer or taxation at time t is $T_t = (\gamma - 1)M_t$. Let ϕ_t denote the value of money in terms of goods in the third subperiod. To simplify the notation, we use the index x_{+1} and x_{-1} to represent a variable corresponding to the next period and the last period, respectively. We focus on stationary equilibria of the economy, which means that the real value of money is always constant. To be more specific, $\phi M = \phi_{+1} M_{+1}$, which implies $\frac{\phi}{\phi_{+1}} = \frac{M_{+1}}{M} = \gamma$.

2.3 Insurance company

There are perfectly competitive insurance companies selling annuities in the third subperiod. They receive incomes from selling annuities to invest in the investment technology in the third subperiod of period t and obtain real returns certainly in the third subperiod of period $t+1$.[5] Assume that insurance companies have a technology to track annuitants. If the annuitant is alive in the second subperiod of $t+1$, they would receive bearer claims as annuity incomes from insurance companies. We assume that insurance companies have a commitment technology so that they will not default on the redemption of claims. The holder of claims could exchange claims for payoffs from the issuing insurance companies in the third subperiod of $t+1$. *Figure* 3 shows the timeline of an insurance company.

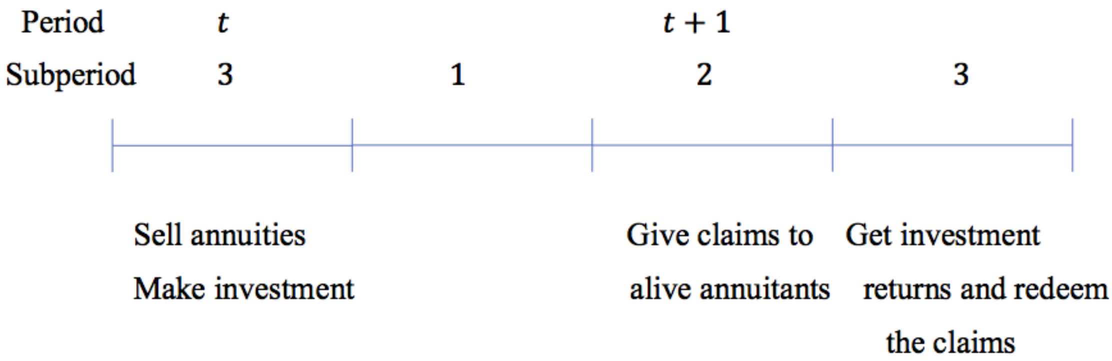

Figure 3: Timeline of insurance companies

The annuity has a main difference against money: receiving the payout amount when the annuitant is alive. The annuity is perfectly divisible. Suppose each unit of annuity is sold at ϕ_t goods. In the second subperiod of $t+1$, insurance companies repay claims worth $(1+i_a)$ unit of money per unit annuity to alive annuitants. We assume that claims mature at the third subperiod of period $t+1$. Specifically, any holder of claims must redeem money from insurance companies in the third

[5]There are risk-based capital requirements for insurers in the United States. In this model, the investment return are certain, so the default probability is zero.

subperiod of period $t+1$, or claims would lose efficacy.[6]

Following Freeman and Kydland (2000) and Li (2011), we assume insurance companies take annuity incomes to invest in the investment technology with a constant rate of return, R. Assume that the investment technology is only accessible to insurance companies. Investing one unit of goods in the technology would turn into $R \geq 1$ goods in the next third subperiod.[7] In addition, insurance companies incur a management cost when holders of claims redeem their payoff. Define that each annuity redeemed in period t incurs the cost of $\phi\theta(A_{-1})$ goods, where A_{-1} is the number of annuities sold by companies in period $t-1$. We assume $\theta(0) = 0$, $\theta'(A_{-1}) > 0$, and $\theta''(A_{-1}) \geq 0$.

Insurance companies have zero profit because of the perfectly competitive market setting. If the return on investment is higher than the return on holding money, which means $R > \frac{1}{\gamma}$, insurance companies invest all incomes in the technology. In this case, insurance companies sell A_{-1} unit of annuities, invest $\phi_{-1}A_{-1}$ unit of goods to get real return worth $R\phi_{-1}A_{-1}$ goods in the third subperiod of next period. If the return on investment is lower than the return on holding money, which means $R \leq \frac{1}{\gamma}$, insurance companies would hold all income as money. Without loss of generality, we consider $R \geq \frac{1}{\gamma}$ in the following discussions. There are only ρA_{-1} unit of annuities being redeemed, so the total management cost is $\rho A_{-1}\phi\theta(A_{-1})$ goods. We derive the payment rate from the zero-profit condition, which is

$$\rho\phi(1+i_a)A_{-1} = R\phi_{-1}A_{-1} - \rho A_{-1}\phi\theta(A_{-1}). \tag{1}$$

The left-hand side of (1) represents the total value of claims in terms of goods which will be redeemed; the right-hand side is the fund of the insurance company

[6]Insurance companies receive investment returns in the third subperiod, so they have to issue claims in the second subperiod. We assume insurance companies have commitment technology and the investment technology is riskless, they will not default. In this case, buyers use claims as means of payment and leave them as bequests and sellers will accept claims in transactions since holders of claims can always redeem them back in the third subperiod.

[7]We could have assumed that the investment rate of return was a function of real inputs; however, it would not affect the main result.

which could be used to repay claims. If $i_a \leq 0$, no agent would purchase the annuity; *i.e.*, $A_{-1} = 0$. The existence of active insurance companies implies that the payout rate of the annuity should be positive; *i.e.*, $i_a > 0$. For our purpose, we consider only equilibrium when $i_a \geq 0$. From (1), the payout rate of the annuity when $A_{-1} > 0$ is

$$1 + i_a = \frac{\gamma R}{\rho} - \theta(A_{-1}). \tag{2}$$

The annuity payment rate, i_a, is affected by the inflation rate, γ, the survival rate of buyers, ρ, the return on the investment technology, R, and the management cost, $\theta(A_{-1})$.

Because the utility function is quasi-linear in the third subperiod and agents are born and live for two to four subperiods, this model delineates an economy resembling the life-cycle model with representative agents having the bequest motive and incorporate insurance companies while keeping distribution of asset portfolio analytically tractable.

Before we proceed, we review the timing of events and trading arrangement. In the first subperiod of period t, buyers spend their money holding to buy goods. At the beginning of the second subperiod, buyers face the survival shock. Buyers who are alive receive claims as annuity incomes from insurance companies and they use money left from the first subperiod and claims in exchange for goods. Sellers in the third subperiod redeem the claims that they received from buyers in the second subperiod. All agents determine their portfolios of money and the annuity in the third subperiod of $t-1$ where everyone can produce and consume. New born sellers enter in the first subperiod and new born buyers enter in the second subperiod or the third subperiod.

3 Optimal choices for sellers and buyers

We focus on stationary equilibria. Let $W_i(m_i, a_i)$ with a portfolio (m_i, a_i) and $W_j(m_j, b_j)$ with a portfolio (m_j, b_j) denote the expected value function of a buyer, and $W_i^s(m_i^s, a_i^s)$ with a portfolio (m_i^s, a_i^s) and $W_j^s(m_j^s, b_j^s)$ with a portfolio (m_j^s, b_j^s) denote the expected value function of a seller when entering subperiod i or j, where $i = 1$, and $j = 2,3$. Denote that b_j is the number of claims that a buyer holds and b_j^s is the number of claims that a seller holds when entering subperiod j, where $j = 2,3$. Notice that a_1 is the number of annuities bought by a buyer and a_1^s is the number of annuities bought by a seller. Besides, b_2 is the number of claims received by a buyer from insurance companies when he is alive so that $b_2 = a_1$, b_3 is the the remaining number of claims as bequests by a buyer after consuming in the second subperiod, and b_j^s is the number of claims received by a seller through exchange, where $j = 2,3$.

3.1 Seller's decision

Sellers are born in the first subperiod of period t, but insurance companies sell annuities in the third subperiod of period $t-1$. Moreover, since sellers die certainly at the end of the third subperiod of period t, it is impossible for them and their heirs to receive annuity incomes in period $t+1$. In this case, sellers have no incentive to buy the annuity in the third subperiod of period t, which implies $a_1^s = 0$. Sellers would not receive claims in first-subperiod transactions, hence $b_2^s = 0$.

3.1.1 The third subperiod

Sellers only consume in the third subperiod where they could produce as well. A seller carries m_3^s money, redeems claims for $(1+i_a)b_3^s$ units of money, produces h^s goods and consumes X^s goods in the third subperiod.[8] Since sellers would not buy any annuity, they only have to determine the amount of money, $m_{1,+1}^s$, bequeathed to heirs. The value function of a seller holding portfolio (m_3^s, b_3^s) to

[8] A seller produces in the first subperiod in exchange for money and produces in the second subperiod in exchange for money and annuity claims.

the third subperiod is

$$\begin{aligned} W_3^s(m_3^s, b_3^s) &= \max_{X^s, h^s, m_{1,+1}^s} \left\{ U(X^s) - h^s + \beta W_{1,+1}^s(m_{1,+1}^s) \right\} \\ \text{s.t. } X^s &= h^s + \phi[m_3^s + (1+i_a)b_3^s] - \phi m_{1,+1}^s. \end{aligned} \tag{3}$$

Define that $z^s = \phi[m_3^s + (1+i_a)b_3^s]$ is the real value of the portfolio held by a seller when entering the third subperiod. Substitute h^s from the budget constraint and rearrange (3):

$$W_3^s(z^s) = z^s + \max_{X^s, m_{1,+1}^s} \left\{ U(X^s) - X^s - \phi m_{1,+1}^s + \beta W_{1,+1}^s(m_{1,+1}^s) \right\}.$$

Define

$$W_3^s(0) = \max_{X^s, m_{1,+1}^s} \left\{ U(X^s) - X^s - \phi m_{1,+1}^s + \beta W_{1,+1}^s(m_{1,+1}^s) \right\},$$

and we get

$$W_3^s(z^s) = z^s + W_3^s(0). \tag{4}$$

This is the linearity of $W_3^s(z^s)$. Since the utility function is quasi-linear, the seller's bequest decision, $m_{1,+1}^s$, is irrelevant to the value of z^s. The first-order conditions are

$$U'(X^s) = 1, \tag{5}$$

$$\phi \geq \beta W_{1m,+1}^s(m_{1,+1}^s), \ = \text{ if } m_{1,+1}^s > 0, \tag{6}$$

where $W_{1m,+1}^s(m_{1,+1}^s)$ is the marginal utility of an additional unit of money as bequest to the child. We get $X^s = X^*$ from (5), which implies X^s reaches the first-best level. The envelope conditions are

$$W_{3m}^s = \frac{\partial W_3^s(z^s)}{\partial m_3^s} = \phi, \tag{7}$$

$$W_{3b}^s = \frac{\partial W_3^s(z^s)}{\partial b_3^s} = \phi(1+i_a). \tag{8}$$

3.1.2 The first and second subperiod

Let p_i denote the nominal price of q_i in terms of money, i=1,2. A new-born seller produces in the first two subperiods and all revenues will be left to consume in the third subperiod. Therefore a new-born seller in the first subperiod solves

$$W_1^s(m_1^s) = \max_{q_1^s, q_2^s} \left\{ -c_1(q_1^s) - \beta c_2(q_2^s) + \beta^2 W_3^s(\phi(m_1^s + p_1 q_1^s + p_2 q_2^s)) \right\}. \tag{9}$$

After applying the linearity property, the seller's problem becomes

$$W_1^s(m_1^s) = \max_{q_1^s, q_2^s} \left\{ -c_1(q_1^s) - \beta c_2(q_2^s) + \beta^2 \left[\phi(m_1^s + p_1 q_1^s + p_2 q_2^s) + W_3^s(0) \right] \right\}.$$

The solution (q_1^s, q_2^s) satisfies

$$c_1'(q_1^s) = \beta^2 \phi p_1, \tag{10}$$

$$c_2'(q_2^s) = \beta \phi p_2. \tag{11}$$

In this case, we derive the nominal prices of goods from (10) and (11) in the first and second subperiod, where

$$p_1 = \frac{c_1'(q_1^s)}{\beta^2 \phi}, \tag{12}$$

$$p_2 = \frac{c_2'(q_2^s)}{\beta \phi}. \tag{13}$$

The envelope condition is

$$W_{1m}^s = \frac{\partial W_1^s(m_1^s)}{\partial m_1^s} = \beta^2 \phi. \tag{14}$$

Plug (14) into (6) and we get

$$\phi \geq \beta^3 \phi_{+1}. \tag{15}$$

Conditions (12) and (13) show that sellers should be indifferent producing in three subperiods. Bequeathing or not is indifferent to sellers since the cost equals the benefit of leaving bequest, which is $\gamma = \beta^3$. If $\gamma > \beta^3$, sellers would not bequeath their children, hence $m^s_{1,+1} = 0$. Condition (15) also shows that the inflation rate has a lower bound, which is $\gamma \geq \beta^3$.

3.2 Buyer's decision

In contrast with sellers, each buyer faces an idiosyncratic survival shock. To determine the behavior of buyers, we start from the third subperiod, in which they start their lives.

3.2.1 The third subperiod

A new-born buyer inherits m_3 money and b_3 claims in the third subperiod. The bequest is worth $m_3 + (1 + i_a)b_3$ unit of money. A new-born buyer produces h goods, consumes X goods, and chooses the portfolio $(m_{1,+1}, a_{1,+1})$ carried to the next period. He solves the following problem:

$$\begin{aligned} W_3(m_3, b_3) &= \max_{X,h,m_{1,+1},a_{1,+1}} \{U(X) - h + \beta W_{1,+1}(m_{1,+1}, a_{1,+1})\} \\ \text{s.t. } x &= h + \phi[m_3 + (1 + i_a)b_3] + \phi T - \phi m_{1,+1} - \phi a_{1,+1}. \end{aligned} \tag{16}$$

Define that $z = \phi[m_3 + (1 + i_a)b_3]$ is the real value of the bequest inherited by a buyer. Substitute h from the budget constraint and rearrange (16):

$$W_3(z) = z + \max_{X,m_{1,+1},a_{1,+1}} \{U(X) - X + \phi T - \phi m_{1,+1} - \phi a_{1,+1} + \beta W_{1,+1}(m_{1,+1}, a_{1,+1})\}.$$

Therefore

$$W_3(z) = z + W_3(0), \tag{17}$$

where

$$W_3(0) = \max_{X,m_{1,+1},a_{1,+1}} \{U(X) - X + \phi T - \phi m_{1,+1} - \phi a_{1,+1} + \beta W_{1,+1}(m_{1,+1}, a_{1,+1})\}.$$

This is the linearity of $W_3(z)$. Since the utility function is quasi-linear, the buyers' portfolio decision, $(m_{1,+1}, a_{1,+1})$, is irrelevant to the value of z. The first-order conditions are

$$U'(X) = 1, \tag{18}$$

$$\phi \geq \beta W_{1m,+1}(m_{1,+1}, a_{1,+1}), \ = \text{ if } m_{1,+1} > 0, \tag{19}$$

$$\phi \geq \beta W_{1a,+1}(m_{1,+1}, a_{1,+1}), \ = \text{ if } a_{1,+1} > 0, \tag{20}$$

where $W_{1m,+1}(m_{1,+1}, a_{1,+1})$ and $W_{1a,+1}(m_{1,+1}, a_{1,+1})$ are the marginal utility of an additional unit of money and annuity taken into the first subperiod of next period respectively. We get $X = X^*$ from (18). Note that new-born buyers possess different portfolios at the beginning of the third subperiod because their parents might be dead or alive in the second subperiod. If a buyer's parent dies at the beginning of the second subperiod, the bequest left to the buyer is $(m_3, 0)$; if a buyer's parent is alive in the second subperiod, the bequest is (m_3, b_3). The first-order conditions (19) and (20) guarantee that the portfolio choice of $(m_{1,+1}, a_{1,+1})$ is irrelevant to the value of third subperiod consumption, X. In this case, the distribution of the asset portfolio of buyers is degenerate at the beginning of each period. The envelope conditions are

$$W_{3m} = \frac{\partial W_3(z)}{\partial m_3} = \phi, \tag{21}$$

$$W_{3b} = \frac{\partial W_3(z)}{\partial b_3} = \phi(1 + i_a). \tag{22}$$

3.2.2 The first and second subperiod

A buyer entering the first subperiod has the portfolio, (m_1, a_1), which is used to finance consumption in the first two subperiods. The buyer's expected lifetime value function of the first subperiod is

$$W_1(m_1, a_1) = \max_{q_1, q_2}\{u_1(q_1) + \rho\beta\left[u_2(q_2) + \beta_a W_3\left(\phi(m_1 - p_1q_1 + (1+i_a)a_1 - p_2q_2)\right)\right] \\ + (1-\rho)\beta\beta_a W_3(\phi(m_1 - p_1q_1))\} \tag{23}$$

$$\text{s.t. } \phi p_1 q_1 \leq \phi m_1 \tag{24}$$

$$\phi p_2 q_2 \leq \phi[m_1 - p_1q_1 + (1+i_a)a_1]. \tag{25}$$

A buyer faces a survival shock at the beginning of the second subperiod. With probability ρ, the buyer is alive. He receives annuity payoffs, consumes q_2, and leaves $m_1 - p_1q_1 + (1+i_a)a_1 - p_2q_2$ to his offspring. With probability $1-\rho$, the buyer is dead and leaves $m_1 - p_1q_1$ units of money to his offspring. Rearrange (23) by making use of the linearity property of W_3:

$$W_1(m_1, a_1) = \max_{q_1, q_2}\{u_1(q_1) + \rho\beta u_2(q_2) + \beta\beta_a\phi(m_1 - p_1q_1) \\ + \rho\beta\beta_a\phi[(1+i_a)a_1 - p_2q_2] + \beta\beta_a W_3(0)\}.$$

We apply the Lagrange multiplier method to solve the buyer's problem. Let λ_1 and λ_2 denote the Lagrange multipliers of budget constraints (24) and (25), respectively, in the first and second subperiod. The Lagrange function is

$$\begin{aligned}\mathcal{L} =& u_1(q_1) + \rho\beta u_2(q_2) + \beta\beta_a\phi(m_1 - p_1q_1) \\ &+ \rho\beta\beta_a\phi[(1+i_a)a_1 - p_2q_2] + \beta\beta_a W_3(0) \\ &+ \lambda_1\phi(m_1 - p_1q_1) + \lambda_2\phi(m_1 - p_1q_1 + (1+i_a)a_1 - p_2q_2).\end{aligned} \tag{26}$$

The first-order conditions are

$$u_1'(q_1) = \phi p_1(\beta\beta_a + \lambda_1 + \lambda_2), \tag{27}$$

$$\rho\beta u_2'(q_2) = \phi p_2(\rho\beta\beta_a + \lambda_2). \tag{28}$$

The complementary slackness conditions are

$$\lambda_1 \phi(m_1 - p_1 q_1) = 0, \tag{29}$$

$$\lambda_2 \phi(m_1 - p_1 q_1 + (1 + i_a)a_1 - p_2 q_2) = 0. \tag{30}$$

Condition (27) clarifies that consuming additional q_1 tightens the constraint of the second subperiod and the constraint of the first subperiod simultaneously. This is because the second-subperiod budget constraint (25) depends on the residual money left from the first subperiod. Condition (29) implies that if the buyer does not spend all money in the first subperiod, we get $\lambda_1 = 0$.

For the completeness of the model, the market clearing conditions of consumption goods in first two subperiods are

$$q_1 = q_1^s, \tag{31}$$

$$\rho q_2 = q_2^s. \tag{32}$$

The market clearing condition of the annuity is

$$a_1 = A_{-1}. \tag{33}$$

Other than the case of $\gamma = \beta^3$, sellers never bequeath their heirs. Without loss of generality, we assume that money is held by buyers only. The market clearing condition of money is

$$m_1 = M_{-1}. \tag{34}$$

3.3 The optimal portfolio choice for buyers

In order to determine the optimal portfolio, we derive the marginal value of the value function in terms of money and the annuity. Plug conditions (27) and (28) into the envelope conditions, they can be written as [9]

$$W_{1m} = \phi(\beta\beta_a + \lambda_1 + \lambda_2), \tag{35}$$

$$W_{1a} = \phi(1 + i_a)(\rho\beta\beta_a + \lambda_2). \tag{36}$$

Holding an additional unit of money helps a buyer loosen constraints in the first two subperiods and leave more bequests to his child. Holding an additional unit of annuity benefits a buyer by loosening the constraint in the second subperiod and bequeathing his child conditional upon being alive. An asset has two dimensions: return and liquidity.[10] Specifically, liquidity of an asset in this economy includes its ability to finance the buyer's first subperiod consumption, the second subperiod consumption, and child's consumption.[11]

A buyer must use money to finance his consumption in the first subperiod since $u_1'(0) \to \infty$, which leads to $m_1 > 0$. In contrast, a buyer could use money or annuity incomes to finance consumption in the second subperiod. A stationary equilibrium satisfies $\frac{\phi_{-1}}{\phi} = \gamma$. We lag one period of (19) and (20) to combine with conditions (35) and (36). Then, we obtain

$$\lambda_1 + \lambda_2 = \frac{\gamma - \beta^2\beta_a}{\beta}, \tag{37}$$

$$\lambda_2 \leq \frac{\gamma - \rho(1 + i_a)\beta^2\beta_a}{\beta(1 + i_a)}, \ = \text{ if } a_1 > 0. \tag{38}$$

Condition (38) shows that if total benefit of annuities on relaxing the constraint, $\phi\beta(1 + i_a)\lambda_2$, and that of leaving bequests, $\phi\rho(1 + i_a)\beta^2\beta_a$, is not higher than the cost ϕ_{-1}, there is no demand for the annuity. Combine (12), (13), (27), (28), (37),

[9]See the proof in Appendix A.

[10]Although the ex-ante return of annuity is uncertain, we use the term "return" on annuity to refer to the amount of annuity payout.

[11]"[L]iquidity–the extent to which an asset can facilitate exchange as a means of payment..." (Li et al. (2012)) defines liquidity. In our context, liquidity refers to the same context, which is the ability of an asset to exchange consumption for buyers when they are young and old, and for their descendants.

and (38), we obtain

$$\frac{1}{\beta^2}(\lambda_1+\lambda_2)=\frac{u_1'(q_1)}{c_1'(q_1)}-\frac{\beta_a}{\beta}=\frac{\gamma-\beta^2\beta_a}{\beta^3}, \tag{39}$$

$$\frac{1}{\rho\beta^2}\lambda_2=\frac{u_2'(q_2)}{c_2'(\rho q_2)}-\frac{\beta_a}{\beta}\leq\frac{\gamma-\rho(1+i_a)\beta^2\beta_a}{\rho(1+i_a)\beta^3},\ =\text{ if } a_1>0. \tag{40}$$

After simplifying, we have

$$\frac{u_1'(q_1)}{c_1'(q_1)}=\frac{\gamma}{\beta^3}, \tag{41}$$

$$\frac{u_2'(q_2)}{c_2'(\rho q_2)}\leq\frac{\gamma}{\rho(1+i_a)\beta^3},\ =\text{ if } a_1>0. \tag{42}$$

Condition (41) implies that buyers carry money to the point at which the marginal benefit of an addition unit of money, $\beta^3\phi\frac{u_1'(q_1)}{c_1'(q_1)}$, equals the marginal cost, ϕ_{-1}.[12] Condition (42) implies that if buyers have the annuity demand, they would purchase up to the point at which the marginal benefit of an additional unit of annuity, $\beta^3\phi\rho\frac{(1+i_a)u_2'(q_2)}{c_2'(\rho q_2)}$, equals the marginal cost, ϕ_{-1}.[13] If the marginal cost of the annuity is always larger than the marginal benefit, they have no demand for the annuity.

4 Equilibrium

Definition 1. *A stationary equilibrium is a list of value functions* (W_i, W_i^s), *buyers' choice* $(X, h, q_1, q_2, m_i, a_1, b_2, b_3)$, *sellers' choice* $(X^s, h^s, q_1^s, q_2^s, m_i^s, a_1^s, b_2^s, b_3^s)$, *where* $i = 1, 2, 3$, *insurance companies' annuity sales volume* A_{-1}, *prices* (p_1, p_2), *a sequence of*

[12]The procedure is similar in Li and Li (2013). Given the market price p_1, a buyer with an additional unit of money buys $\frac{1}{p_1}$ units of q_1, which generates utility $\frac{u_1'(q_1)}{p_1}$. The additional money costs ϕ_{-1}. To compare the benefit and cost at the same time point, we discount the benefit one subperiod and get $\beta\frac{u_1'(q_1)}{p_1}=\phi_{-1}$. After converting p_1 to $\frac{c_1'(q_1)}{\beta^2\phi}$ from (12), it becomes $\beta^3\phi\frac{u_1'(q_1)}{c_1'(q_1)}=\phi_{-1}$, which implies $\frac{u_1'(q_1)}{c_1'(q_1)}=\frac{\gamma}{\beta^3}$.

[13]Given the market price p_2, a buyer with an additional unit of annuity could buy $\frac{1+i_a}{p_1}$ units of q_2 when he is alive, which generates expected utility $\rho\frac{(1+i_a)u_2'(q_2)}{p_2}$. The additional annuity costs ϕ_{-1}. To compare the benefit and cost at the same time point, we discount the benefit two subperiod and get $\beta^2\rho\frac{(1+i_a)u_2'(q_2)}{p_2}=\phi_{-1}$. After converting p_2 to $\frac{c_2'(\rho q_2)}{\beta\phi}$ from (13), it becomes $\beta^3\phi\rho\frac{(1+i_a)u_2'(q_2)}{c_2'(\rho q_2)}=\phi_{-1}$, which implies $\frac{u_2'(q_2)}{c_2'(\rho q_2)}=\frac{\gamma}{\rho(1+i_a)\beta^3}$.

money value $\{\phi_t\}$*, and the payout rate* i_a *that solve* (3), (9), (16), *and* (23) *given the inability of sellers to purchase the annuity and satisfy market clearing conditions* (31)–(34), *constant real value of money, and insurance companies' zero profit condition* (2).

The tightness of constraints are displayed through λ_1 and λ_2. We focus on possible stationary equilibria categorized through the tightness of constraints and the availability of annuities, which are unconstrained equilibrium with ($\lambda_1 = 0, \lambda_2 = 0$), full annuitization equilibrium with ($\lambda_1 > 0, \lambda_2 = 0$), full annuitization equilibrium with ($\lambda_1 > 0, \lambda_2 > 0$), partial annuitization equilibrium with ($\lambda_1 = 0, \lambda_2 > 0$), and pure cash equilibrium with ($\lambda_1 = 0, \lambda_2 > 0$). Since the consumption in the first subperiod could only be financed by money, we will focus on the means of payment decision in the second subperiod. For the analysis, we specify the lower bound of the inflation rate, γ, in the following lemma.

Lemma 1. *In a stationary equilibrium,* $\gamma \geq \max\{\beta^2\beta_a, \beta^3, \rho(1+i_a)\beta^2\beta_a\}$.

Lemma 1 results from (15), (37), and (38). It prevents buyers and sellers from accumulating money and the annuity infinitely.

4.1 Unconstrained equilibrium

We begin by considering the unconstrained equilibrium, in which constraints in the first and second subperiod are unbinding; *i.e.*, $\lambda_1 = \lambda_2 = 0$.

Proposition 1. *An unconstrained stationary equilibrium exists if and only if* $\gamma = \beta^2\beta_a$. *Under an unconstrained stationary equilibrium, it achieves the first-best allocation,* $(q_1, q_2) = (q_1^*, q_2^*)$*, if and only if* $\beta_a = \beta$.[14]

proof. See Appendix A. □

The difference of the altruism degree between sellers and buyers is a factor of inefficiency.From (12) and (13), sellers with higher time preference are willing

[14]Under the more general setup on discount factors such that we reserve notations β_1, β_2, β_a, β_a^s, β_2^s, and β_3^s, the condition would be $\beta_2\beta_a = \beta_2^s\beta_3^s$. Under the assumption of $\beta_2 = \beta_2^s = \beta$, the condition becomes $\beta_a = \beta_3^s$. It is clearer that the key factor is how much a seller cares about himself in the third subperiod. The logic is stated in the main text.

to sell goods only under a higher price since consumption in the third subperiod is less valuable. If β is smaller, prices on q_1 and q_2 are higher, hence less consumption.

Proposition 2. *Suppose $\beta_a \geq \beta$. Under an unconstrained equilibrium:*

1. *If $a_1 > 0$, (q_1, q_2) solves*

$$\frac{u_1'(q_1)}{c_1'(q_1)} = \frac{\beta_a}{\beta},$$
$$\frac{u_2'(q_2)}{c_2'(\rho q_2)} = \frac{\beta_a}{\beta}.$$

The portfolio (m_1, a_1) solves

$$\frac{\beta^2 \beta_a R}{\rho} - \theta(a_1) = \frac{1}{\rho},$$
$$m_1 = M_{-1}.$$

The payout rate, i_a, satisfies

$$1 + i_a = \frac{1}{\rho}.$$

2. *If $a_1 = 0$, (q_1, q_2) solves*

$$\frac{u_1'(q_1)}{c_1'(q_1)} = \frac{\beta_a}{\beta},$$
$$\frac{u_2'(q_2)}{c_2'(\rho q_2)} = \frac{\beta_a}{\beta}.$$

The portfolio (m_1, a_1) solves

$$m_1 = M_{-1},$$
$$a_1 = 0.$$

The payout rate, i_a, satisfies

$$i_a = 0.$$

Under an unconstrained equilibrium, agents are indifferent to finance consumption through money and the annuity in the second subperiod under $a_1 > 0$. Money and the annuity could not loosen constraints in the first two subperiods; therefore the ability of leaving bequests to offsprings remains to be the only consideration.

Lemma 2. *In an unconstrained equilibrium with $a_1 > 0$, $\rho(1 + i_a) = 1$.*

The condition $\rho(1+i_a) = 1$ shows that the return of the annuity, $(1+i_a)$, offsets its lower liquidity which is caused by ρ in financing children's consumption. The *ex ante* expected return from the annuity equals the return of money. In this situation, it is indifferent to bequeath through the annuity or money. If $\rho(1+i_a) < 1$, the annuity is dominated by money, which causes the annuity to vanish in the economy and $i_a = 0$.

Proposition 3. *Under an unconstrained equilibrium with $a_1 > 0$, if the survival rate, ρ, increases, the payout rate, i_a, and the demand for the annuity, a_1, decrease.*

proof. See Appendix A. □

Given the rate of return of the investment technology, higher survival rate implies that there are more annuitants split the return, hence less payout amount for annuitants. Lower payout amount decreases the annuity demand while higher survival rate makes the annuity more attractive. In equilibrium, $\rho(1 + i_a) = 1$ holds, so the decreasing number of annuities results from the zero-profit condition of insurance companies when ρ increases.

4.2 Constrained equilibrium

In a constrained equilibrium, at least one resource constraint in the first and the second subperiod is binding. From lemma 1 and proposition 1, we have $\gamma > \beta^2\beta_a$.

We categorize constrained equilibria by the portfolio choice into full annuitization, partial annuitization, and pure cash equilibrium.

4.2.1 Full annuitization equilibrium

A full annuitization equilibrium depicts the situation that all consumption in the second subperiod, which can be interpreted as the after-retirement stage in this model, is fully financed by annuity income. In this equilibrium, q_1 is financed by money only, and q_2 is financed by the annuity. Since all money is used in the first subperiod, the constraint in the first subperiod is binding, which means $\lambda_1 > 0$. In this case, buyers must finance consumption in the second subperiod through annuities. Given the fact of the coexistence of money and the annuity, we characterize the full annuitization equilibrium in the following proposition.

Proposition 4. *Suppose $\gamma > \beta^2\beta_a$. In a full annuitization equilibrium:*

1. *If $\lambda_2 = 0$, the payout rate, i_a, solves*

$$\rho(1 + i_a)\beta^2\beta_a = \gamma.$$

The quantity (q_1, q_2) solves

$$\frac{u_1'(q_1)}{c_1'(q_1)} = \frac{\gamma}{\beta^3},$$
$$\frac{u_2'(q_2)}{c_2'(\rho q_2)} = \frac{\beta_a}{\beta}.$$

The price ϕ and the portfolio (m_1, a_1) solves

$$\frac{c_1'(q_1)}{\beta^2} q_1 = \phi m_1,$$
$$m_1 = M_{-1},$$
$$1 + i_a = \frac{\gamma R}{\rho} - \theta(a_1).$$

2. *If $\lambda_2 > 0$, the price ϕ, the payout rate, i_a, the quantity (q_1, q_2), and the portfolio*

(m_1, a_1) *solve*

$$
\begin{aligned}
&\frac{u_1'(q_1)}{c_1'(q_1)} = \frac{\gamma}{\beta^3},\\
&\frac{u_2'(q_2)}{c_2'(\rho q_2)} = \frac{\gamma}{\rho(1+i_a)\beta^3},\\
&\frac{c_1'(q_1)}{\beta^2} q_1 = \phi m_1,\\
&\frac{c_2'(\rho q_2)}{\beta} q_2 = \phi(1+i_a)a_1,\\
&m_1 = M_{-1},\\
&1 + i_a = \frac{\gamma R}{\rho} - \theta(a_1).
\end{aligned}
$$

Lemma 3. *In a full annuitization equilibrium,* $\rho(1+i_a) > 1$.

proof. See Appendix A. □

A buyer could have financed consumption in the second subperiod through money and the annuity. In a full annuitization equilibrium, buyers choose to finance consumption in the second subperiod and bequeath through the annuity. This implies that the annuity dominates money after the first subperiod. Lemma 3 states a necessary condition of a full annuitization equilibrium: the expected return of the annuity must be higher than the return of money. It shows that the payout rate has a lower bound: $\frac{1}{\rho}$. The survival rate, ρ, is a source of illiquidity. This implies the rate of return dominance: the return must surpass the illiquidity of the annuity.

Recall $\gamma \geq \rho(1+i_a)\beta^2\beta_a$ from lemma 1, which is equivalent to $\phi_{-1} \geq \phi\rho(1+i_a)\beta^2\beta_a$. The expected return of the annuity is $\phi\rho(1+i_a)\beta^2\beta_a$ with the cost ϕ_{-1}. Under $\gamma = \rho(1+i_a)\beta^2\beta_a$, the cost and the expected return of the annuity are identical. This results in the efficient consumption level through the annuity in the second subperiod, which is $\lambda_2 = 0$. We get $\gamma > \rho(1+i_a)\beta^2\beta_a$ in a full annuitization equilibrium with $\lambda_2 > 0$ from (38). Under $\gamma > \rho(1+i_a)\beta^2\beta_a$, the expected value of the annuity is lower. Thus, buyers are willing to consume less than the efficient level in the second subperiod.

Proposition 5. *Under a full annuitization equilibrium with $\lambda_2 > 0$:*

1. *A change in the inflation rate has real effects in a full annuitization equilbrium:* $\frac{\partial q_1}{\partial \gamma} < 0$, $\frac{\partial q_2}{\partial \gamma} < 0$, $\frac{\partial \phi}{\partial \gamma} < 0$, $\frac{\partial i_a}{\partial \gamma} > 0$, *and* $\frac{\partial \phi(1+i_a)a_1}{\partial \gamma} < 0$ *if R is sufficiently large.*

2. *The altruism degree of buyers does not affect prices and allocations.*

proof. See Appendix A. □

R should be sufficiently large to sustain higher i_a, which is $1 + i_a > \frac{1}{\rho}$ from lemma 3. Higher inflation devaluates money, lowers ϕ and causes q_1 and q_2 to decrease. Higher inflation also raises the payout rate since investment returns of insurance companies are goods, which converts to more money under higher inflation. The decreasing of ϕ and the increasing of i_a generate opposite effects on the annuity demand. Although the impact on the number of annuities is indeterminate, we could infer that the real value of the annuity, $\phi(1 + i_a)a_1$, decreases since annuity incomes are all used to finance q_2 and q_2 decreases. Higher inflation tightens the constraints in the first and the second subperiod. Since q_1 and q_2 decrease at the same time, the social welfare decreases as the inflation rate increases.

A buyer would not bequeath in a full annuitization equilibrium because they run out of money in the first subperiod and annuity incomes in the second subperiod. A change in the altruism degree of buyers would not increase bequests at the margin, so it does not affect prices and allocations.

4.2.2 Partial annuitization equilibrium

A partial annuitization equilibrium refers to the circumstances that consumption in the second subperiod is financed by money and the annuity. Since money will be left to the second subperiod, the constraint in the first subperiod does not bind, which means $\lambda_1 = 0$. The $\lambda_2 = 0$ case is the unconstrained equilibrium, so we focus on the case in which $\lambda_2 > 0$. Since the annuity is active, we know (38) and (40) hold at equality. From (37), we obtain $\lambda_2 = \frac{\gamma - \beta^2 \beta_a}{\beta}$. We characterize the paritial-annuitization equilibrium as follows.

Proposition 6. *Suppose $\gamma > \beta^2 \beta_a$. In a partial annuitization equilibrium, in which $\lambda_1 = 0$, $\lambda_2 > 0$ with the annuity being active, (q_1, q_2) solves*

$$\frac{u_1'(q_1)}{c_1'(q_1)} = \frac{\gamma}{\beta^3},$$
$$\rho\left[\frac{u_2'(q_2)}{c_2'(\rho q_2)} - \frac{\beta_a}{\beta}\right] = \frac{\gamma - \beta^2 \beta_a}{\beta^3}.$$

The price ϕ, the payout rate, i_a, and the portfolio (m_1, a_1) solve

$$\frac{c_1'(q_1)}{\beta^2} q_1 + \frac{c_2'(\rho q_2)}{\beta} q_2 = \phi[m_1 + (1 + i_a) a_1],$$
$$m_1 = M_{-1},$$
$$1 + i_a = \frac{\gamma R}{\rho} - \theta(a_1),$$
$$1 + i_a = \frac{\gamma}{\gamma - (1 - \rho)\beta^2 \beta_a}.$$

A buyer carries money and the annuity after the third subperiod, which means total benefits of both assets are identical since they all cost ϕ_{-1}. In contrast to the full annuitization economy, here money helps nothing in loosening the constraint in the first subperiod; therefore money and the annuity compete in the following aspects: returns, ability in financing q_2, and ability in financing consumption of offsprings. The unbinding constraint in the first subperiod implies that the benefit of money left for q_2 and bequests equals the benefit of consuming q_1 marginally. When a buyer is alive, he could finance consumption in the second subperiod and bequeath through both money and the annuity. When a buyer is dead, he could not consume, where money and the annuity are equally useless in terms of financing q_2. Moreover, if a buyer is dead, he could not obtain annuity incomes, so money has an additional function as bequests while the annuity lose efficacy, which is the relative illiquidity for the annuity. Money is a relatively liquid asset while the annuity is a relatively high return asset. In a partial annuitization equilibrium, the higher return of the annuity in financing q_2 and financing children's consumption must compensate for the contingent bequest under probability ρ. The relative return offsets the relative liquidity between the annuity and money.

Lemma 4. *In a partial annuitization equilibrium,* $\rho(1 + i_a) < 1$.

proof. See Appendix A. $\square$

Since money lacks the ability of loosening the first-subperiod constraint, the payout rate need not be large in which the relative return offsets the relative illiquidity of the annuity. Lemma 4 restricts the payout rate of the annuity so that the benefit of the annuity would not overtake the benefit of money.

Proposition 7. *Under a partial annuitization equilibrium,*

1. *A change in the inflation rate has real effects in a partial annuitization equilibrium:* $\frac{\partial q_1}{\partial \gamma} < 0$, $\frac{\partial q_2}{\partial \gamma} < 0$, $\frac{\partial a_1}{\partial \gamma} > 0$, $\frac{\partial \phi}{\partial \gamma} < 0$, *and* $\frac{\partial i_a}{\partial \gamma} < 0$.
2. *A change in the degree of altruism of buyers has real effects in a partial annuitization equilbrium:* $\frac{\partial q_1}{\partial \beta_a} = 0$, $\frac{\partial q_2}{\partial \beta_a} > 0$, $\frac{\partial a_1}{\partial \beta_a} < 0$, $\frac{\partial \phi}{\partial \beta_a}$ *is indeterminate, and* $\frac{\partial i_a}{\partial \beta_a} > 0$.

proof. See Appendix A. $\square$

Higher inflation devaluates money, in which buyers consume less in q_1 and q_2. Higher inflation also raises the payout rate at first because insurance companies could exchange investment returns for more money. The annuity serves as an inflation-protected asset to some degree. In this case, the demand for the annuity increases. The annuity is more attractive and money becomes less valuable.

Unlike a full annuitization equilibrium, the degree of altruism stands in a niche of liquidity under a partial annuitization equilibrium. The higher degree of altruism of buyers makes the relative illiquidity of the annuity against money more severely because there are bequests left to the child when a buyer dies.

Buyers have incentives to bequeath more, so they are willing to leave more money after the first subperiod. If a buyer is alive, he could consume more in the second subperiod from additional money originating from the higher bequest motive, so q_2 increases. This also results in the decreasing demand for the annuity and the decreasing management cost, which leads to the higher payout rate.

Since money would be left to the second subperiod, there are bequests left to offsprings if a buyer dies in a partial annuitization equilibrium. Returns and liquidity play key roles in determining the demand for money and the annuity. Even if annuities repay higher payout amount than money in the second subperiod, the uncertainty to be left as bequests might offset the benefit of higher returns. This is the reason that buyers have less, or almost zero annuity demand. It is illiquidity that crowds out the annuity demand, even the annuity always has the dominant return over money.

4.2.3 Pure cash equilibrium

If the rate of return of the annuity is sufficiently low, buyers may not demand the annuity, and the annuity will vanish in the economy. Then, buyers finance consumption in the first and the second subperiod all by money. Since money will be left to the second subperiod, the constraint in the first subperiod never binds, which means $\lambda_1 = 0$. We characterize the pure cash economy with $\lambda_2 > 0$ as follows.

Proposition 8. *Suppose $\gamma > \beta^2\beta_a$. In a pure cash equilibrium with $\lambda_1 = 0$ and $\lambda_2 > 0$, (q_1, q_2) solves*

$$
\begin{aligned}
&\frac{u_1'(q_1)}{c_1'(q_1)} = \frac{\gamma}{\beta^3}, \\
&\rho\left[\frac{u_2'(q_2)}{c_2'(\rho q_2)} - \frac{\beta_a}{\beta}\right] = \frac{\gamma - \beta^2\beta_a}{\beta^3}.
\end{aligned}
$$

The price ϕ and the portfolio $(m_1, a_1) = (m_1, 0)$ solves

$$\frac{c_1'(q_1)}{\beta^2} q_1 + \frac{c_2'(\rho q_2)}{\beta} q_2 = \phi m_1,$$
$$m_1 = M_{-1}.$$

The payout rate

$$i_a = 0.$$

In the pure cash economy, the relative illiquidity dominates the relative return of the annuity against money. The certainty of money to be left to descendants outweighs the higher return of the annuity. The reason for the zero annuity demand is its inability to bequeath.

Proposition 9. *If a buyer does not have the bequest motive, which means $\beta_a = 0$, the pure cash equilibrium would not exist.*

proof. See Appendix A. □

The intuition is that money and the annuity could both finance consumption when the buyer is alive, which implies the liquidity in financing q_2 is identical. With no bequest motive, even though money could be left as bequests, it plays no role for buyers. Since it is only the return that matters for buyers in consuming q_2, the annuity with positive payout rate dominates money in the second subperiod. In this case, the factor of illiquidity of the annuity disappears, so it travels back to the rate of return dominance situation.

5 Numerical example

We use numerical examples to help us observe more implications about full annuitization equilibrium and partial annuitization equilibrium.

Figure 4 shows the effect of inflation rate on consumption, the price, the payout rate, and the annuity demand under a full annuitization equilibrium.[15] The result is consistent with the analysis in proposition 5. The effects of the inflation rate on ϕ, q_1, and q_2 are negative because the higher inflation lowers the value of money, which causes decrease in q_1 and q_2. Although the increase or decrease of the annuity depends on the functional form of $u_1(q_1)$, $u_2(q_2)$, $c_1(q_1^s)$, $c_2(q_2^s)$, and $\theta(A)$, the key point is that the value of the annuity as a means of payment in the second subperiod decreases. Notice that the inflation rate may not be too small since the payout rate would decrease. If the payout rate reaches zero lower bound, a full annuitization equilibrium does not exist.

The following numerical results display that the annuitization rate is less than 1 percent, which is consistent with the empirical finding in Johnson et al. (2004). We show the effect of the inflation rate and the degree of altruism on consumption, the price, the payout rate, the annuity demand, and the annuitization rate under a partial annuitization equilibrium (see *Figure* 5 and *Figure* 6).[16] Define annuitization rate to be

$$\frac{\phi(1+i_a)a_1}{\phi p_2 q_2},$$

which denotes the fraction of the consumption q_2 financed by the annuity. Empirically, the annuitization rate refers to the share of household income for adults ages 65 and older made up by the private annuity. In this paper, money serves as the unique means of payment in the first subperiod, $\frac{a_1}{m_1+a_1}$ might dilute the value since a fraction of m_1 is not for the retirement stage.[17] If a buyer is alive, he would use all money and annuity incomes to consume in the second subperiod

[15]In *Figure* 4, the functional forms are $u_1(q_1) = \frac{q_1^{1-\sigma_1}}{1-\sigma_1}$, $u_2(q_2) = \eta\frac{q_2^{1-\sigma_2}}{1-\sigma_2}$, $c_1(q_1^s) = \frac{(q_1^s)^{\epsilon_1}}{\epsilon_1}$, $c_2(q_2^s) = \frac{(q_2^s)^{\epsilon_2}}{\epsilon_2}$, and $\theta(A) = \delta A^{\epsilon}$. The parameter values are $\sigma_1 = 0.2$, $\sigma_2 = 0.2$, $\epsilon_1 = 1$, $\epsilon_2 = 1$, $\epsilon = 1$, $\delta = 0.0005$, $\eta = 1$, $R = 1.03$, $M_{-1} = 100$, $\beta = 0.97$, $\beta_a = 0.9$, $\gamma = 1.016$, and $\rho = 0.95$.

[16]In *Figure* 5 and *Figure* 6, the functional forms are $u_1(q_1) = \frac{q_1^{1-\sigma_1}}{1-\sigma_1}$, $u_2(q_2) = \eta\frac{q_2^{1-\sigma_2}}{1-\sigma_2}$, $c_1(q_1^s) = \frac{(q_1^s)^{\epsilon_1}}{\epsilon_1}$, $c_2(q_2^s) = \frac{(q_2^s)^{\epsilon_2}}{\epsilon_2}$, and $\theta(A) = \delta A^{\epsilon}$. The parameter values are $\sigma_1 = 0.2$, $\sigma_2 = 0.2$, $\epsilon_1 = 1$, $\epsilon_2 = 1$, $\epsilon = 1$, $\delta = 1$, $\eta = 0.5$, $R = 1.03$, $M_{-1} = 100$, $\beta = 0.97$, $\beta_a = 0.3$, $\gamma = 1.016$, and $\rho = 0.9447094$. We adopt some parameters which are widely used in saving literatures.

[17]For the robustness check, if the annuitization rate is defined as $\frac{a_1}{m_1+a_1}$, the analysis is qualitatively consistent with $\frac{\phi(1+i_a)a_1}{\phi p_2 q_2}$.

since $\lambda_2 > 0$; *i.e.*, the total value of consumption q_2 equals the total wealth at the beginning of the second subperiod. To correspond to the index used in empirical data, we adopt $\frac{\phi(1+i_a)a_1}{\phi p_2 q_2}$.

In the partial annuitization equilibrium, the inflation rate affects relative return while the degree of altruism affects relative liquidity. *Figure* 5 shows that the effect of the inflation rate on the annuitization rate is positive because the higher inflation lowers the value of money, ϕ, then increases the demand for the annuity, a_1, and raises the annuitization rate. *Figure* 6 shows that the effect of the degree of altruism on the annuitization rate is negative because it makes the annuity illiquid, then reduces the demand for the annuity, a_1, and reduces the annuitization rate. This also implies that money accounts for a bigger share of the retirement wealth when the degree of altruism increases. Different from past literatures, this paper generates the general equilibrium results in which the annuitization rate is almost zero when there is the bequest motive reducing the liquidity of the annuity.

6 Discussion: the effect of inflation

We show that a change in the inflation rate does not affect the allocation and real prices in a life-cycle partial equilibrium economy with bequest motives. Consider a partial equilibrium model in which retirement wealth, the annuity price and the annuity payoff are exogenous. For simplicity, assume that an agent lives at most two periods after retirement. He faces a survival shock ρ, which is the survival rate, after the first period and dies certainly at the end of the second period. Define that $\pi = \frac{\rho y}{(1+r)(1-\lambda)}$ is the real value that the agent uses to buy an annuity in which λ is the load from Lockwood (2012) and y is the real annuity income. Suppose a representative agent maximizes his expected utility after retirement:

$$u(c_1) + (1-\rho)v(e_1) + \rho\beta[u(c_2) + v(e_2)]$$

subject to

$$
\begin{aligned}
e_1 &= w - \pi - c_1 \\
e_2 &= (1+r)(w - \pi - c_1) - (c_2 - y),
\end{aligned}
$$

where $v(e)$ is utility from e real bequests, r is the real return of an asset, β is the discount rate, and w is real retirement wealth. Since we are going to discuss the inflation rate, we need to inject nominal prices into the economy: p_1 and p_2. The bequests become

$$
\begin{aligned}
p_1 e_1 &= p_1(w - \pi - c_1) \\
p_2 e_2 &= \frac{p_2}{p_1}(1+r)p_1(w - \pi - c_1) - p_2(c_2 - y).
\end{aligned}
$$

From the Fisher equation, we have nominal interest rate equals

$$
\frac{p_2}{p_1}(1+r) = \gamma(1+r),
$$

where γ is the inflation rate. We observe that the inflation rate does not affect the real value of bequests since the nominal interest rate grows in proportion to the inflation rate. Suppose there is a government in this economy that raises the inflation rate, it would not affect any real variable. In this case, the inflation does not have real effect in life-cycle partial equilibrium economy.

However, in this paper, insurance companies own a real investment technology. This implies that the inflation rate has different effects on the returns of money and the annuity. As long as insurance companies invest fractionally in real return subjects (*e.g.*, land), the inflation rate will have real effects in the economy. If there are frictions that make inflation have different effects on the returns of money and nominal assets, such as bonds, the inflation rate would generate different effects on the returns of money and the annuity even if insurance companies invest totally in nominal assets. Therefore inflation affects allocation and prices of the economy in this paper.

7 Conclusion

In this paper, we point out that the key factor of the low annuity demand is relative illiquidity of the annuity compared with other assets. Many literatures rely on the medical expense risk, *e.g.*, Ameriks et al. (2011) and Peijnenburg et al. (2017), to explain why people would not annuitize all wealth, but they fail to explain why people annuitize petty wealth or do not annuitize any wealth. We set aside everything but the survival risk and the bequest motive to generate the result that the return and liquidity are endogenous. It is the illiquidity of the annuity which originates from the bequest motive that significantly reduce the demand for the annuity.

Different from Lockwood (2012) in which the annuity price and a fraction of pre-existing annuitization wealth are exogenous, the monetary search general equilibrium model in which prices and quantities are endogenously determined gives us a comprehensively analytical framework. Although this model is simplified, we are able to clarify each significant characteristic and each mechanism. The benefits and drawbacks of money and the annuity are clearly characterized. The suggestion for future research is that we could analyze the effects on policies (*e.g.*, the annuity loan, the inheritance tax, and the deferred tax on purchasing annuities), which could be extended in a general equilibrium model of this paper.

This paper has three main contributions. First, it generates a general equilibrium model that specifies behaviors of agents and insurance companies clearly, in which almost no demand for the annuity could be derived under the bequest motive with the endogenous annuity payout rate. Second, this paper generates the analytical form, not only numerical solutions, so that it is able to do the qualitative analysis in an implicit function form structure. Third, injecting a nominal asset allows us to discuss the effect of the inflation rate on allocation and prices.

Piling up with numerical examples, this paper sheds light on resolving the annuity puzzle.

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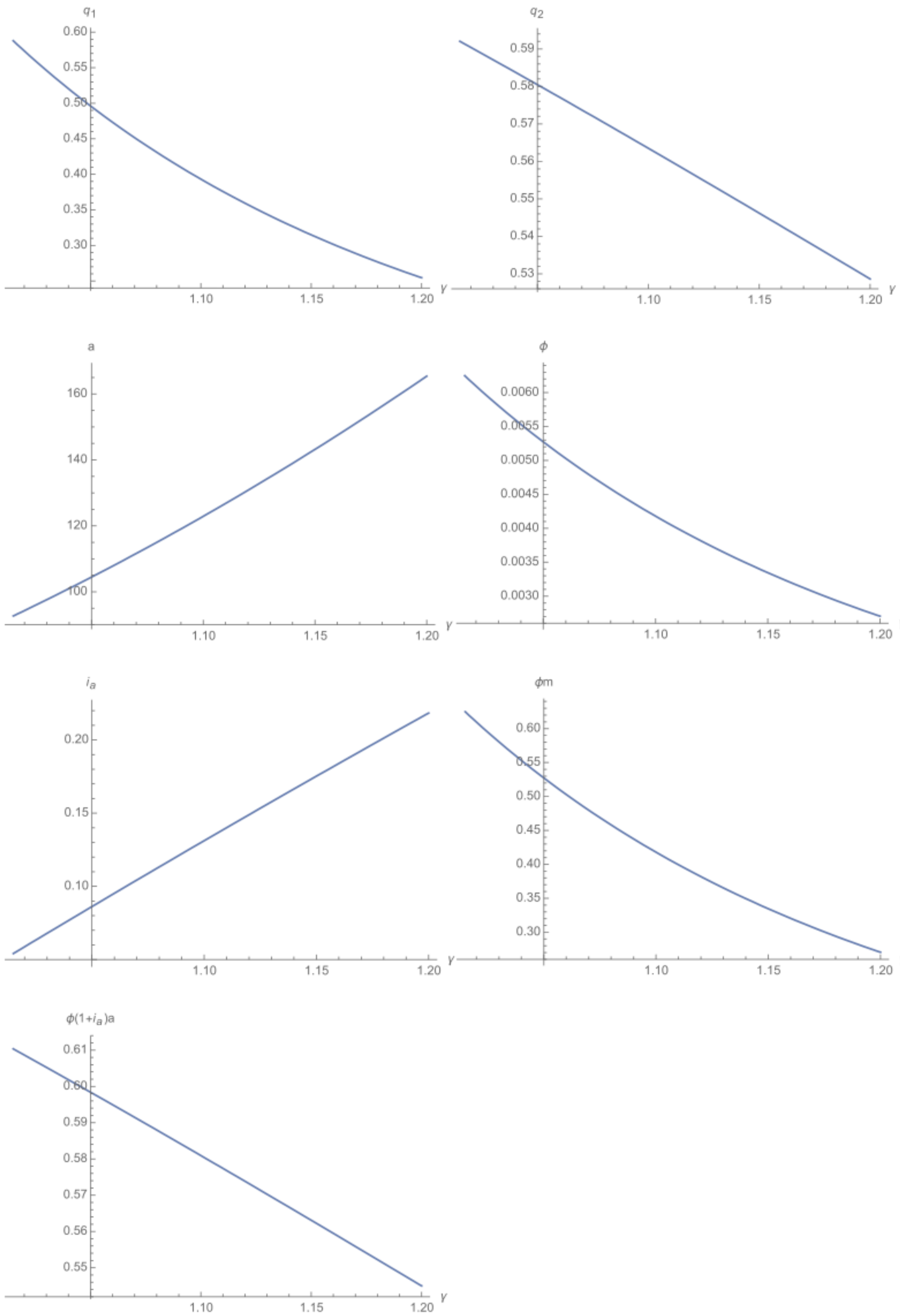

Figure 4: The effect of inflation rate in the full annuitization equilibrium

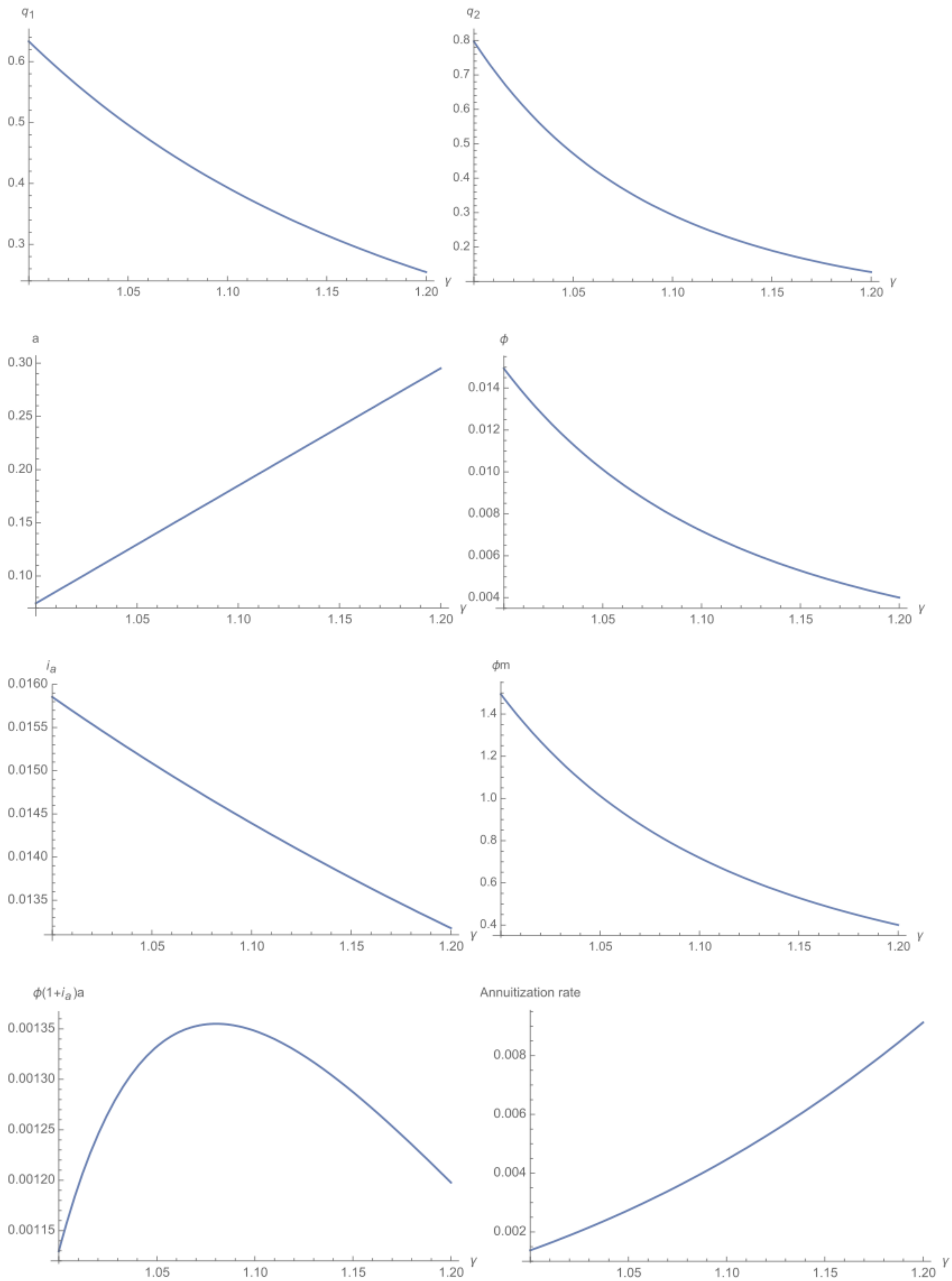

Figure 5: The effect of inflation rate in the partial annuitization equilibrium

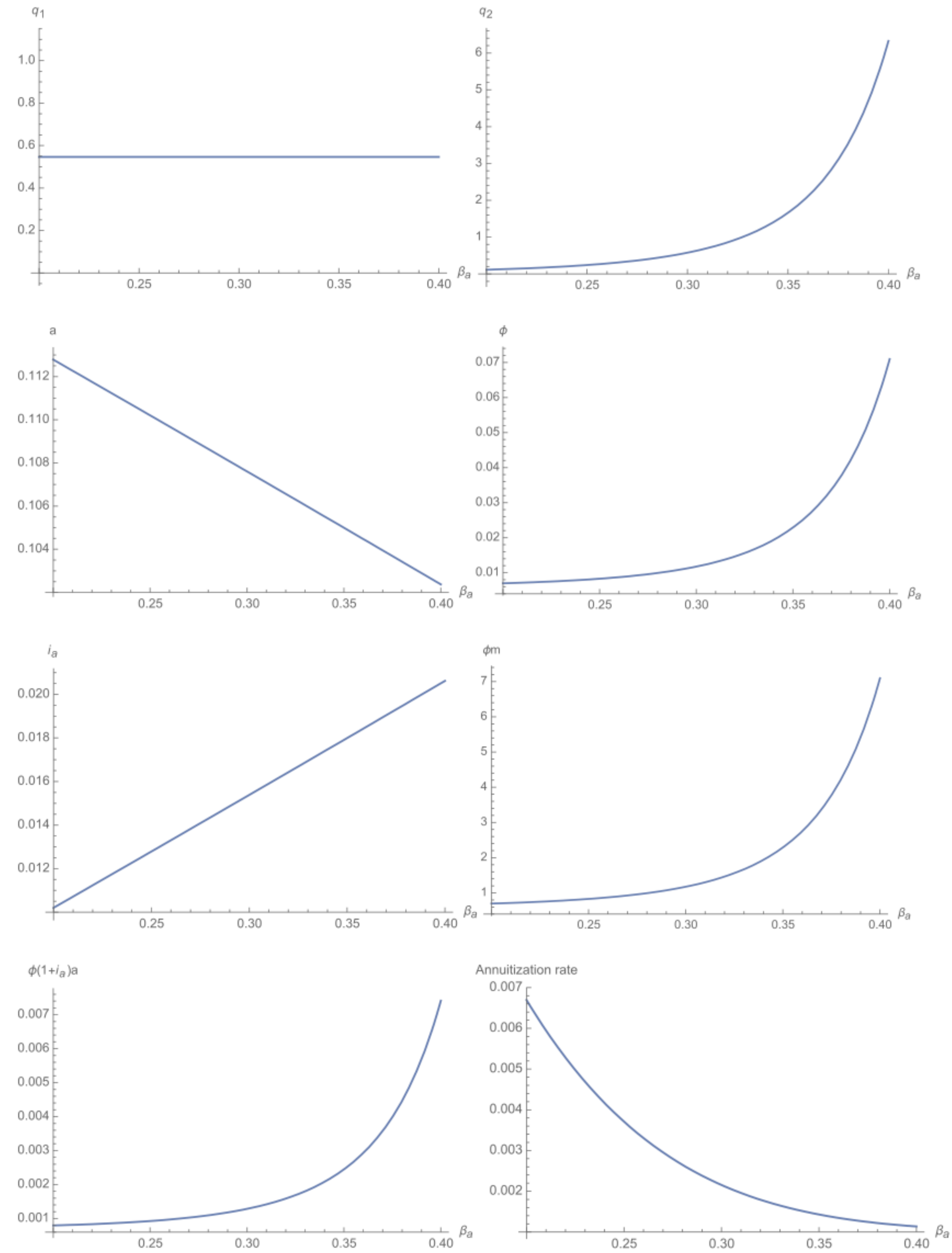

Figure 6: The effect of degree of altruism in the partial annuitization equilibrium

Appendix A

Deviration of envelope conditions in the first subperiod

Differential equation (23) with respect to m_1 and a_1, we get

$$\begin{aligned}
W_{1m} =& \beta\beta_a\phi + \{u_1'(q_1) - \beta\beta_a\phi p_1\}\frac{\partial q_1}{\partial m_1} + [\rho\beta u_2'(q_2) - \rho\beta\beta_a\phi p_2]\frac{\partial q_2}{\partial m_1} \\
=& \beta\beta_a\phi + \phi p_1(\lambda_1 + \lambda_2)\frac{\partial q_1}{\partial m_1} + \phi p_2\lambda_2\frac{\partial q_2}{\partial m_1} \\
=& \beta\beta_a\phi + \phi\lambda_1 p_1\frac{\partial q_1}{\partial m_1} + \phi\lambda_2\left(p_1\frac{\partial q_1}{\partial m_1} + p_2\frac{\partial q_2}{\partial m_1}\right), \\
W_{1a} =& \rho\beta\beta_a\phi(1+i_a) + \{u_1'(q_1) - \beta\beta_a\phi p_1\}\frac{\partial q_1}{\partial a_1} + [\rho\beta u_2'(q_2) - \rho\beta\beta_a\phi p_2]\frac{\partial q_2}{\partial a_1} \\
=& \rho\beta\beta_a\phi(1+i_a) + \phi p_1(\lambda_1 + \lambda_2)\frac{\partial q_1}{\partial a_1} + \phi p_2\lambda_2\frac{\partial q_2}{\partial a_1} \\
=& \rho\beta\beta_a\phi(1+i_a) + \phi\lambda_1 p_1\frac{\partial q_1}{\partial a_1} + \phi\lambda_2\left(p_1\frac{\partial q_1}{\partial m_1} + p_2\frac{\partial q_2}{\partial a_1}\right).
\end{aligned}$$

Now we only have to worry about the case of $\lambda_1 > 0$ and $\lambda_2 > 0$. If $\lambda_1 > 0$, it implies that $m_1 = p_1 q_1$. We differentiate both sides with respect to m_1 and a_1 to get $\frac{\partial q_1}{\partial m_1} = \frac{1}{p_1}$ and $\frac{\partial q_1}{\partial a_1} = 0$ respectively. If $\lambda_2 > 0$, it implies that $m_1 + (1+i_a)a_1 = p_1q_1 + p_2q_2$. Differentiate both sides with respect to m_1 and a_1 to get $1 = p_1\frac{\partial q_1}{\partial m_1} + p_2\frac{\partial q_2}{\partial m_1}$ and $(1+i_a) = p_1\frac{\partial q_1}{\partial a_1} + p_2\frac{\partial q_2}{\partial a_1}$. Then we obtain equation (35) and (36):

$$\begin{aligned}
W_{1m} &= \phi(\beta\beta_a + \lambda_1 + \lambda_2), \\
W_{1a} &= \phi(1+i_a)(\rho\beta\beta_a + \lambda_2).
\end{aligned}$$

Proposition 1

proof. The first statement is simply the result of (39). Suppose there is an unconstrained equilibrium, which means $\lambda_1 = \lambda_2 = 0$. From (39), we have $\gamma = \beta^2\beta_a$. Suppose $\gamma = \beta^2\beta_a$, it is obvious that $\lambda_1 = \lambda_2 = 0$ since $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$. The first statement is proved.

Under an unconstrained equilibrium, we have $\gamma = \beta^2\beta_a$ and $\lambda_1 = \lambda_2 = 0$ from the statement above. It guarantees that $\beta_a \geq \beta$ from *lemma 1*. In this case,

$\frac{u_1'(q_1)}{c_1'(q_1)} = \frac{u_2'(q_2)}{c_2'(\rho q_2)} = \frac{\beta_a}{\beta} \geq 1$. Suppose $q_1 = q_1^*$ and $q_2 = q_2^*$, we get $\frac{u_1'(q_1)}{c_1'(q_1)} = \frac{u_2'(q_2)}{c_2'(\rho q_2)} = 1$, which means $\beta_a = \beta$. Suppose $\beta_a = \beta$, we have $\frac{u_1'(q_1)}{c_1'(q_1)} = \frac{u_2'(q_2)}{c_2'(\rho q_2)} = 1$, which means $q_1 = q_1^*$ and $q_2 = q_2^*$. □

Proposition 3

proof. From proposition 2, we have $\frac{\partial i_a}{\partial \rho} = \frac{-1}{\rho^2} < 0$ and $\frac{\gamma R - 1}{\rho} = \theta(a_1)$. Since $a_1 > 0$, it guarantees that $\gamma R - 1 > 0$. It is obvious that $\frac{\partial a_1}{\partial \rho} < 0$. □

Lemma 3

proof. If $\lambda_2 = 0$, the proof is complete from lemma 1, proposition 1.

If $\lambda_2 > 0$, from (37) and equality in (38), we have

$$\begin{aligned}
\lambda_1 &= \frac{\gamma - \beta^2\beta_a}{\beta} - \frac{\gamma - \rho(1+i_a)\beta^2\beta_a}{\beta(1+i_a)} > 0 \\
\Rightarrow 1 + i_a &> \frac{\gamma}{\gamma - (1-\rho)\beta^2\beta_a} > \frac{\beta^2\beta_a}{\beta^2\beta_a - (1-\rho)\beta^2\beta_a} = \frac{1}{\rho}.
\end{aligned}$$

□

Proposition 5

proof. In a full annuitization equilibrium, $(q_1, q_2, a_1, \phi, i_a)$ satisfies proposition 4. Define

$$\begin{aligned}
f_1(q_1, q_2, a_1, \phi, i_a; \gamma) &= \frac{u_1'(q_1)}{c_1'(q_1)} - \frac{\gamma}{\beta^3}; \\
f_2(q_1, q_2, a_1, \phi, i_a; \gamma) &= \frac{u_2'(q_2)}{c_2'(\rho q_2)} - \frac{\gamma}{\rho(1+i_a)\beta^3}; \\
f_3(q_1, q_2, a_1, \phi, i_a; \gamma) &= \frac{c_1'(q_1)q_1}{\beta^2} - \phi M_{-1}; \\
f_4(q_1, q_2, a_1, \phi, i_a; \gamma) &= \frac{c_2'(\rho q_2)q_2}{\beta} - \phi(1+i_a)a_1; \\
f_5(q_1, q_2, a_1, \phi, i_a; \gamma) &= 1 + i_a - \frac{\gamma R}{\rho} + \theta(a_1).
\end{aligned}$$

Let f_{ix} denote $\frac{\partial f_i}{\partial x}$, where $i = 1,2,3,4,5$ and $x = q_1, q_2, a_1, \phi, i_a, \gamma$. We have

$$f_{1q_1} = \frac{u_1''(q_1)c_1'(q_1) - c_1''(q_1)u_1'(q_1)}{[c_1'(q_1)]^2} < 0, f_{1\gamma} = \frac{-1}{\beta^3} < 0;$$
$$f_{2q_2} = \frac{u_2''(q_2)c_2'(\rho q_2) - \rho c_2''(\rho q_2)u_2'(q_2)}{[c_2'(\rho q_2)]^2} < 0, f_{2i} = \frac{\gamma}{\rho(1+i_a)^2\beta^3} > 0, f_{2\gamma} = \frac{-1}{\rho(1+i_a)\beta^3} < 0;$$
$$f_{3q_1} = \frac{c_1''(q_1)q_1 + c_1'(q_1)}{\beta^2} > 0, f_{3\phi} = -M_{-1} < 0;$$
$$f_{4q_2} = \frac{\rho c_2''(\rho q_2)q_2 + c_2'(\rho q_2)}{\beta} > 0, f_{4a_1} = -\phi(1+i_a) < 0, f_{4\phi} = -(1+i_a)a_1 < 0, f_{4i_a} = -\phi a_1 < 0;$$
$$f_{5a_1} = \theta'(a_1) > 0, f5i_a = 1 > 0, f_{5\gamma} = \frac{-R}{\rho} < 0;$$
$$f_{1q_2} = f_{1a_1} = f_{1\phi} = f_{1i_a} = f_{2q_1} = f_{2a_1} = f_{2\phi} = f_{3q_2} = f_{3a_1} = f_{3i_a} = f_{4q_1} = f_{5q_1} = f_{5q_2} = f_{5\phi} = 0.$$

We assume that R is large enough to satisfy

$$f_{4a_1}f_{5i_a} - f_{4i_a}f_{5a_1} = \phi\left[a_1\theta'(a_1) - \frac{\gamma R}{\rho} + \theta(a_1)\right] < 0,$$
$$-f_{2\gamma}(f_{4a_1}f_{5i_a} - f_{4i_a}f_{5a_1}) + f_{5\gamma}f_{2i_a}f_{4a_1} < 0.$$

We rewrite the equilibrium condition into the linear system:

$$\begin{bmatrix} f_{1q_1} & 0 & 0 & 0 & 0 \\ 0 & f_{2q_2} & 0 & 0 & f_{2i_a} \\ f_{3q_1} & 0 & 0 & f_{3\phi} & 0 \\ 0 & f_{4q_2} & f_{4a_1} & f_{4\phi} & f_{4i_a} \\ 0 & 0 & f_{5a_1} & 0 & f_{5i_a} \end{bmatrix}\begin{bmatrix} dq_1 \\ dq_2 \\ da_1 \\ d\phi \\ di_a \end{bmatrix} = -\begin{bmatrix} f_{1\gamma}d\gamma \\ f_{2\gamma}d\gamma \\ 0 \\ 0 \\ f_{5\gamma}d\gamma \end{bmatrix}.$$

Let Λ^{FA}, $\Lambda^{FA}_{1\gamma}$, $\Lambda^{FA}_{2\gamma}$, $\Lambda^{FA}_{3\gamma}$, $\Lambda^{FA}_{4\gamma}$, $\Lambda^{FA}_{5\gamma}$ denote the determinants of the following matrices:

$$\Lambda^{FA}=\begin{vmatrix} f_{1q_1} & 0 & 0 & 0 & 0 \\ 0 & f_{2q_2} & 0 & 0 & f_{2i_a} \\ f_{3q_1} & 0 & 0 & f_{3\phi} & 0 \\ 0 & f_{4q_2} & f_{4a_1} & f_{4\phi} & f_{4i_a} \\ 0 & 0 & f_{5a_1} & 0 & f_{5i_a} \end{vmatrix}, \Lambda^{FA}_{1\gamma}=\begin{vmatrix} -f_{1\gamma} & 0 & 0 & 0 & 0 \\ -f_{2\gamma} & f_{2q_2} & 0 & 0 & f_{2i_a} \\ 0 & 0 & 0 & f_{3\phi} & 0 \\ 0 & f_{4q_2} & f_{4a_1} & f_{4\phi} & f_{4i_a} \\ -f_{5\gamma} & 0 & f_{5a_1} & 0 & f_{5i_a}, \end{vmatrix},$$

$$\Lambda^{FA}_{2\gamma}=\begin{vmatrix} f_{1q_1} & -f_{1\gamma} & 0 & 0 & 0 \\ 0 & -f_{2\gamma} & 0 & 0 & f_{2i_a} \\ f_{3q_1} & 0 & 0 & f_{3\phi} & 0 \\ 0 & 0 & f_{4a_1} & f_{4\phi} & f_{4i_a} \\ 0 & -f_{5\gamma} & f_{5a_1} & 0 & f_{5i_a} \end{vmatrix}, \Lambda^{FA}_{3\gamma}=\begin{vmatrix} f_{1q_1} & 0 & -f_{1\gamma} & 0 & 0 \\ 0 & f_{2q_2} & -f_{2\gamma} & 0 & f_{2i_a} \\ f_{3q_1} & 0 & 0 & f_{3\phi} & 0 \\ 0 & f_{4q_2} & 0 & f_{4\phi} & f_{4i_a} \\ 0 & 0 & -f_{5\gamma} & 0 & f_{5i_a} \end{vmatrix},$$

$$\Lambda^{FA}_{4\gamma}=\begin{vmatrix} f_{1q_1} & 0 & 0 & -f_{1\gamma} & 0 \\ 0 & f_{2q_2} & 0 & -f_{2\gamma} & f_{2i_a} \\ f_{3q_1} & 0 & 0 & 0 & 0 \\ 0 & f_{4q_2} & f_{4a_1} & 0 & f_{4i_a} \\ 0 & 0 & f_{5a_1} & -f_{5\gamma} & f_{5i_a} \end{vmatrix}, \Lambda^{FA}_{5\gamma}=\begin{vmatrix} f_{1q_1} & 0 & 0 & 0 & -f_{1\gamma} \\ 0 & f_{2q_2} & 0 & 0 & -f_{22\gamma} \\ f_{3q_1} & 0 & 0 & f_{3\phi} & 0 \\ 0 & f_{4q_2} & f_{4a_1} & f_{4\phi} & 0 \\ 0 & 0 & f_{5a_1} & 0 & -f_{5\gamma} \end{vmatrix}.$$

After calculating, we have

$$\begin{aligned}
\Lambda^{FA} &= -f_{1q_1}f_{3\phi}[f_{2q_2}(f_{4a_1}f_{5i_a}-f_{4i_a}f_{5a_1})+f_{4q_2}f_{2i_a}f_{5a_1}],\\
\Lambda^{FA}_{1\gamma} &= f_{1\gamma}f_{3\phi}[f_{2q_2}(f_{4a_1}f_{5i_a}-f_{4i_a}f_{5a_1})+f_{4q_2}f_{2i_a}f_{5a_1}],\\
\Lambda^{FA}_{2\gamma} &= -f_{1q_1}f_{3\phi}[-f_{2\gamma}(f_{4a_1}f_{5i_a}-f_{4i_a}f_{5a_1})+f_{5\gamma}f_{2i_a}f_{4a_1}]+f_{1\gamma}f_{3q_1}f_{2i_a}f_{4\phi}f_{5a_1},\\
\Lambda^{FA}_{3\gamma} &= -f_{1q_1}f_{3\phi}[f_{2q_2}f_{4i_a}f_{5\gamma}-f_{4q_2}(-f_{2\gamma}f_{5i_a}+f_{2i_a}f_{5\gamma})]+f_{1\gamma}f_{3q_1}f_{4\phi}f_{2q_2}f_{5i_a},\\
\Lambda^{FA}_{4\gamma} &= -f_{3q_1}f_{1\gamma}[f_{2q_2}(f_{4a_1}f_{5i_a}-f_{4i_a}f_{5a_1})+f_{4q_2}f_{2i_a}f_{5a_1}],\\
\Lambda^{FA}_{5\gamma} &= f_{1q_1}f_{3\phi}(f_{2q_2}f_{4a_1}f_{5\gamma}+f_{2\gamma}f_{4q_2}f_{5a_1})-f_{1\gamma}f_{3q_1}f_{2q_2}f_{4\phi}f_{5a_1},
\end{aligned}$$

and $\Lambda^{FA}<0$, $\Lambda^{FA}_{1\gamma}>0$, $\Lambda^{FA}_{2\gamma}>0$, $\Lambda^{FA}_{3\gamma}$ is indeterminate, $\Lambda^{FA}_{4\gamma}>0$, $\Lambda^{FA}_{5\gamma}<0$. Therefore $\frac{\partial q_1}{\partial\gamma}=\frac{\Lambda^{FA}_{1\gamma}}{\Lambda^{FA}}<0$, $\frac{\partial q_2}{\partial\gamma}=\frac{\Lambda^{FA}_{2\gamma}}{\Lambda^{FA}}<0$, $\frac{\partial\phi}{\partial\gamma}=\frac{\Lambda^{FA}_{4\gamma}}{\Lambda^{FA}}<0$, $\frac{\partial i_a}{\partial\gamma}=\frac{\Lambda^{FA}_{5\gamma}}{\Lambda^{FA}}>0$. Since q_2 decreases, we get $\phi(1+i_a)a_1$ decreases.

Observe that β_a does not show in the equilibrium conditions. Thus, a change in β_a does not have real effects on the equilibrium prices and allocations. $\square$

Lemma 4

proof. Equation (35) and (36) are identical with $\lambda_1 = 0$:

$$\phi(\beta\beta_a + \lambda_2) = \phi(1+i_a)(\rho\beta\beta_a + \lambda_2).$$

Suppose $\rho(1+i_a) \geq 1$, we have

$$\beta\beta_a + \lambda_2 = (1+i_a)\rho\beta\beta_a + (1+i_a)\lambda_2 > \beta\beta_a + \lambda_2,$$

which is a contradiction. Thus, $\rho(1+i_a) < 1$. $\square$

Proposition 7

proof. In a partial annuitization equilibrium, $(q_1, q_2, a_1, \phi, i_a)$ satisfies proposition 6. Define

$$\begin{aligned}
g_1(q_1, q_2, a_1, \phi, i_a; \gamma, \beta_a) &= \frac{u_1'(q_1)}{c_1'(q_1)} - \frac{\gamma}{\beta^3};\\
g_2(q_1, q_2, a_1, \phi, i_a; \gamma, \beta_a) &= \frac{\rho u_2'(q_2)}{c_2'(\rho q_2)} - \frac{\gamma}{\beta^3} + (1+\rho)\frac{\beta_a}{\beta};\\
g_3(q_1, q_2, a_1, \phi, i_a; \gamma, \beta_a) &= \frac{c_1'(q_1)q_1}{\beta^2} + \frac{c_2'(\rho q_2)q_2}{\beta} - \phi[M_{-1} + (1+i_a)a_1];\\
g_4(q_1, q_2, a_1, \phi, i_a; \gamma, \beta_a) &= 1 + i_a - \frac{\gamma R}{\rho} + \theta(a_1);\\
g_5(q_1, q_2, a_1, \phi, i_a; \gamma, \beta_a) &= 1 + i_a - \frac{\gamma}{\gamma - (1-\rho)\beta^2\beta_a}.
\end{aligned}$$

Let g_{ix} denote $\frac{\partial g_i}{\partial x}$, where $i = 1,2,3,4,5$ and $x = q_1, q_2, a_1, \phi, i_a, \gamma, \beta_a$. We have

$$\begin{aligned}
&g_{1q_1} = \frac{u_1''(q_1)c_1'(q_1) - c_1''(q_1)u_1'(q_1)}{[c_1'(q_1)]^2} < 0, g_{1\gamma} = \frac{-1}{\beta^3} < 0;\\
&g_{2q_2} = \rho\frac{u_2''(q_2)c_2'(\rho q_2) - \rho c_2''(\rho q_2)u_2'(q_2)}{[c_2'(\rho q_2)]^2} < 0, g_{2\gamma} = \frac{-1}{\beta^3} < 0, g_{2\beta_a} = \frac{1-\rho}{\beta} > 0;\\
&g_{3q_1} = \frac{c_1''(q_1)q_1 + c_1'(q_1)}{\beta^2} > 0, g_{3q_2} = \frac{\rho c_2''(\rho q_2)q_2 + c_2'(\rho q_2)}{\beta} > 0, g_{3a_1} = -\phi(1+i_a) < 0,\\
&g_{3\phi} = -[M_{-1} + (1+i_a)a_1] < 0, g_{3i_a} = -\phi i_a < 0;\\
&g_{4a_1} = \theta'(a_1) > 0, g_{4i_a} = 1 > 0, g_{4\gamma} = \frac{-R}{\rho} < 0;\\
&g_{5i_a} = 1 > 0, g_{5\gamma} = \frac{-(1-\rho)\beta^2\beta_a}{[\gamma - (1-\rho)\beta^2\beta_a]^2} < 0, g_{5\beta_a} = \frac{-(1-\rho)\beta^2\gamma}{[\gamma - (1-\rho)\beta^2\beta_a]^2} < 0;\\
&g_{1q_2} = g_{1a_1} = g_{1\phi} = g_{1i_a} = g_{2q_1} = g_{2a_1} = g_{2\phi} = g_{2i_a} = g_{4q_1} = g_{4q_2} = g_{4\phi} = g_{5q_1} = g_{5q_2}\\
&= g_{5a_1} = g_{5\phi} = 0.
\end{aligned}$$

We rewrite the equilibrium condition into the linear system:

$$\begin{bmatrix} g_{1q_1} & 0 & 0 & 0 & 0 \\ 0 & g_{2q_2} & 0 & 0 & 0 \\ g_{3q_1} & g_{3q_2} & g_{3a_1} & g_{3\phi} & g_{3i_a} \\ 0 & 0 & g_{4a_1} & 0 & g_{4i_a} \\ 0 & 0 & 0 & 0 & g_{5i_a} \end{bmatrix} \begin{bmatrix} dq_1 \\ dq_2 \\ da_1 \\ d\phi \\ di_a \end{bmatrix} = -\begin{bmatrix} g_{1\gamma}d\gamma \\ g_{2\gamma}d\gamma + g_{2\beta_a}d\beta_a \\ 0 \\ g_{4\gamma}d\gamma \\ g_{5\gamma}d\gamma + g_{5\beta_a}d\beta_a \end{bmatrix}.$$

Let Λ^{PA}, $\Lambda^{PA}_{1\gamma}$, $\Lambda^{PA}_{2\gamma}$, $\Lambda^{PA}_{3\gamma}$, $\Lambda^{PA}_{4\gamma}$, $\Lambda^{PA}_{5\gamma}$, $\Lambda^{PA}_{1\beta_a}$, $\Lambda^{PA}_{2\beta_a}$, $\Lambda^{PA}_{3\beta_a}$, $\Lambda^{PA}_{4\beta_a}$, $\Lambda^{PA}_{5\beta_a}$ denote the determinants of the following matrices:

$$\Lambda^{PA} = \begin{vmatrix} g_{1q_1} & 0 & 0 & 0 & 0 \\ 0 & g_{2q_2} & 0 & 0 & 0 \\ g_{3q_1} & g_{3q_2} & g_{3a_1} & g_{3\phi} & g_{3i_a} \\ 0 & 0 & g_{4a_1} & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix}, \Lambda^{PA}_{1\gamma} = \begin{vmatrix} -g_{1\gamma} & 0 & 0 & 0 & 0 \\ -g_{2\gamma} & g_{2q_2} & 0 & 0 & 0 \\ 0 & g_{3q_2} & g_{3a_1} & g_{3\phi} & g_{3i_a} \\ -g_{4\gamma} & 0 & g_{4a_1} & 0 & 1 \\ -g_{5\gamma} & 0 & 0 & 0 & 1 \end{vmatrix},$$

$$\Lambda_{2\gamma}^{PA}=\begin{vmatrix} g_{1q_1} & -g_{1\gamma} & 0 & 0 & 0 \\ 0 & -g_{2\gamma} & 0 & 0 & 0 \\ g_{3q_1} & 0 & g_{3a_1} & g_{3\phi} & g_{3i_a} \\ 0 & -g_{4\gamma} & g_{4a_1} & 0 & 1 \\ 0 & -g_{5\gamma} & 0 & 0 & 1 \end{vmatrix}, \Lambda_{3\gamma}^{PA}=\begin{vmatrix} g_{1q_1} & 0 & -g_{1\gamma} & 0 & 0 \\ 0 & g_{2q_2} & -g_{2\gamma} & 0 & 0 \\ g_{3q_1} & g_{3q_2} & 0 & g_{3\phi} & g_{3i_a} \\ 0 & 0 & -g_{4\gamma} & 0 & 1 \\ 0 & 0 & -g_{5\gamma} & 0 & 1 \end{vmatrix},$$

$$\Lambda_{4\gamma}^{PA}=\begin{vmatrix} g_{1q_1} & 0 & 0 & -g_{1\gamma} & 0 \\ 0 & g_{2q_2} & 0 & -g_{2\gamma} & 0 \\ g_{3q_1} & g_{3q_2} & g_{3a_1} & 0 & g_{3i_a} \\ 0 & 0 & g_{4a_1} & -g_{4\gamma} & 1 \\ 0 & 0 & 0 & -g_{5\gamma} & 1 \end{vmatrix}, \Lambda_{5\gamma}^{PA}=\begin{vmatrix} g_{1q_1} & 0 & 0 & 0 & -g_{1\gamma} \\ 0 & g_{2q_2} & 0 & 0 & -g_{2\gamma} \\ g_{3q_1} & g_{3q_2} & g_{3a_1} & g_{3\phi} & 0 \\ 0 & 0 & g_{4a_1} & 0 & -g_{4\gamma} \\ 0 & 0 & 0 & 0 & -g_{5\gamma} \end{vmatrix},$$

$$\Lambda_{1\beta_a}^{PA}=\begin{vmatrix} 0 & 0 & 0 & 0 & 0 \\ -g_{2\beta_a} & g_{2q_2} & 0 & 0 & 0 \\ 0 & g_{3q_2} & g_{3a_1} & g_{3\phi} & g_{3i_a} \\ 0 & 0 & g_{4a_1} & 0 & 1 \\ -g_{5\beta_a} & 0 & 0 & 0 & 1 \end{vmatrix}, \Lambda_{2\beta_a}^{PA}=\begin{vmatrix} g_{1q_1} & 0 & 0 & 0 & 0 \\ 0 & -g_{2\beta_a} & 0 & 0 & 0 \\ g_{3q_1} & 0 & g_{3a_1} & g_{3\phi} & g_{3i_a} \\ 0 & 0 & g_{4a_1} & 0 & 1 \\ 0 & -g_{5\beta_a} & 0 & 0 & 1 \end{vmatrix},$$

$$\Lambda_{3\beta_a}^{PA}=\begin{vmatrix} g_{1q_1} & 0 & 0 & 0 & 0 \\ 0 & g_{2q_2} & -g_{2\beta_a} & 0 & 0 \\ g_{3q_1} & g_{3q_2} & 0 & g_{3\phi} & g_{3i_a} \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -g_{5\beta_a} & 0 & 1 \end{vmatrix}, \Lambda_{4\beta_a}^{PA}=\begin{vmatrix} g_{1q_1} & 0 & 0 & 0 & 0 \\ 0 & g_{2q_2} & 0 & -g_{2\beta_a} & 0 \\ g_{3q_1} & g_{3q_2} & g_{3a_1} & 0 & g_{3i_a} \\ 0 & 0 & g_{4a_1} & 0 & 1 \\ 0 & 0 & 0 & -g_{5\beta_a} & 1 \end{vmatrix},$$

$$\Lambda_{5\beta_a}^{PA}=\begin{vmatrix} g_{1q_1} & 0 & 0 & 0 & 0 \\ 0 & g_{2q_2} & 0 & 0 & -g_{2\beta_a} \\ g_{3q_1} & g_{3q_2} & g_{3a_1} & g_{3\phi} & 0 \\ 0 & 0 & g_{4a_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & -g_{5\beta_a} \end{vmatrix}.$$

After calculating, we have

$$\Lambda^{PA} = -g_{1q_1}g_{2q_2}g_{3\phi}g_{4a_1},$$
$$\Lambda^{PA}_{1\gamma} = g_{1\gamma}g_{2q_2}g_{3\phi}g_{4a_1},$$
$$\Lambda^{PA}_{2\gamma} = g_{2\gamma}g_{3\phi}g_{1q_1}g_{4a_1},$$
$$\Lambda^{PA}_{3\gamma} = -g_{3\phi}g_{1q_1}g_{2q_2}(g_{5\gamma} - g_{4\gamma}),$$
$$\Lambda^{PA}_{4\gamma} = g_{1q_1}[g_{2q_2}(-g_{3a_1}g_{4\gamma} - g_{3i_a}g_{4a_1}g_{5\gamma} + g_{3a_1}g_{5\gamma}) - g_{3q_2}g_{2\gamma}g_{4a_1}] - g_{1\gamma}g_{3q_1}g_{2q_2}g_{4a_1},$$
$$\Lambda^{PA}_{5\gamma} = g_{1q_1}g_{3\phi}g_{2q_2}g_{4a_1}g_{5\gamma};$$
$$\Lambda^{PA}_{1\beta_a} = 0,$$
$$\Lambda^{PA}_{2\beta_a} = g_{1q_1}g_{3\phi}g_{2\beta_a}g_{4a_1},$$
$$\Lambda^{PA}_{3\beta_a} = -g_{3\phi}g_{1q_1}g_{2q_2}g_{5\beta_a},$$
$$\Lambda^{PA}_{4\beta_a} = g_{1q_1}[g_{2q_2}g_{5\beta_a}(g_{3a_1} - g_{3i_a}g_{4a_1}) - g_{3q_2}g_{2\beta_a}g_{4a_1}],$$
$$\Lambda^{PA}_{5\beta_a} = g_{1q_1}g_{3\phi}g_{2q_2}g_{4a_1}g_{5\beta_a}.$$

We get $\Lambda^{PA} > 0$, $\Lambda^{PA}_{1\gamma} < 0$, $\Lambda^{PA}_{2\gamma} < 0$, $\Lambda^{PA}_{3\gamma} > 0$, $\Lambda^{PA}_{4\gamma} < 0$, $\Lambda^{PA}_{5\gamma} < 0$; $\Lambda^{PA}_{1\beta_a} = 0$, $\Lambda^{PA}_{2\beta_a} > 0$, $\Lambda^{PA}_{3\beta_a} < 0$, $\Lambda^{PA}_{4\beta_a}$ indeterminate, $\Lambda^{PA}_{5\beta_a} > 0$.
Therefore $\frac{\partial q_1}{\partial \gamma} = \frac{\Lambda^{PA}_{1\gamma}}{\Lambda^{PA}} < 0$, $\frac{\partial q_2}{\partial \gamma} = \frac{\Lambda^{PA}_{2\gamma}}{\Lambda^{PA}} < 0$, $\frac{\partial a_1}{\partial \gamma} = \frac{\Lambda^{PA}_{3\gamma}}{\Lambda^{PA}} > 0$, $\frac{\partial \phi}{\partial \gamma} = \frac{\Lambda^{PA}_{4\gamma}}{\Lambda^{PA}} < 0$, $\frac{\partial i_a}{\partial \gamma} = \frac{\Lambda^{PA}_{5\gamma}}{\Lambda^{PA}} < 0$; $\frac{\partial q_1}{\partial \beta_a} = \frac{\Lambda^{PA}_{1\beta_a}}{\Lambda^{PA}} = 0$, $\frac{\partial q_2}{\partial \beta_a} = \frac{\Lambda^{PA}_{2\beta_a}}{\Lambda^{PA}} > 0$, $\frac{\partial a_1}{\partial \beta_a} = \frac{\Lambda^{PA}_{3\beta_a}}{\Lambda^{PA}} < 0$, $\frac{\partial i_a}{\partial \beta_a} = \frac{\Lambda^{PA}_{5\beta_a}}{\Lambda^{PA}} > 0$.
We elaborate how to determine the sign of $\Lambda^{PA}_{4\gamma}$ because it is a little bit complicated. It is sufficient to check that $\frac{R}{\rho} - \frac{(1-\rho)\beta^2\beta_a}{[\gamma-(1-\rho)\beta^2\beta_a]^2} \geq 0$. From the equilibrium condition, we have $\frac{\gamma R}{\rho} - \frac{\gamma}{\gamma-(1-\rho)\beta^2\beta_a} - \theta(a_1) = 0$. Differentiate by γ, we have $\frac{R}{\rho} - \frac{(1-\rho)\beta^2\beta_a}{[\gamma-(1-\rho)\beta^2\beta_a]^2} - \theta'(a_1)\frac{\partial a_1}{\partial \gamma} = 0$. Since $\theta'(a_1)\frac{\partial a_1}{\partial \gamma} \geq 0$, and then we get $\frac{R}{\rho} - \frac{(1-\rho)\beta^2\beta_a}{[\gamma-(1-\rho)\beta^2\beta_a]^2} > 0$. □

Proposition 9

proof. We know that $\lambda_1 = 0$ and $a_1 = 0$. From (37), (38), and (40), we obtain $\frac{u_2'(q_2)}{c_2'(\rho q_2)} - \frac{\beta_a}{\beta} = \frac{\gamma-\beta^2\beta_a}{\rho\beta^3} < \frac{\gamma-\rho\beta^2\beta_a}{\rho\beta^3}$. Suppose that $\beta_a = 0$, we get $\frac{\gamma}{\rho\beta^3} < \frac{\gamma}{\rho\beta^3}$, which is a contradiction. The pure cash economy could not be sustained under $\beta_a = 0$. □